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Quiz – April 20, 2004

Approximate $\int_0^1 \cos(x^2)dx$ with an error at most 5×10^{-5}. Justify your answer.

Answer: Recall that

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots .$$

Replace x by x^2 to see that

$$\cos(x^2) = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots .$$

It follows that

$$\int_0^1 \cos(x^2)dx = \int_0^1 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots dx$$

$$= x - \frac{x^5}{2 \cdot 5} + \frac{x^9}{4! \cdot 9} - \frac{x^{13}}{6! \cdot 13} + \frac{x^{17}}{8! \cdot 17} + \dots \Big|_0^1$$

(*) $$= 1 - \frac{1}{2 \cdot 5} + \frac{1}{4! \cdot 9} - \frac{1}{6! \cdot 13} + \frac{1}{8! \cdot 17} + \dots .$$

Apply the alternating series test. The series (*) alternates, the terms in absolute value are decreasing (since the denominators are growing and the numerators stay at 1), and the terms go to zero. The alternating series test tells us that the series (*) converveges and the distance between the sum of the series and the N^{th} partial sum is at most the $(N+1)^{\text{st}}$ term, for all N. Which is the first term which is less than $5 \times 10^{-5} = \frac{1}{20{,}000}$? We notice that

$$6! \cdot 13 < 20,000 < 8! \cdot 17.$$

It follows that

$$\frac{1}{8! \cdot 17} < 5 \times 10^{-5} < \frac{1}{6! \cdot 13};$$

and we conclude that

$$\boxed{1 - \frac{1}{2 \cdot 5} + \frac{1}{4! \cdot 9} - \frac{1}{6! \cdot 13} \text{ approximates } \int_0^1 \cos(x^2)dx \text{ with an error at most } 5 \times 10^{-5}.}$$