PRINT Your Name:___________________________________

Quiz – March 30, 2004

Estimate the error that is made by approximating the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2}$$

by using the sum of the first five terms. Show complete, coherent work.

Answer: The difference between

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2} \quad \text{and} \quad \sum_{k=1}^{5} \frac{1}{1+k^2}$$

is

$$\sum_{k=6}^{\infty} \frac{1}{1+k^2}.$$

The function $f(x) = \frac{1}{1+x^2}$ is positive and decreasing. We can over estimate the value of $\sum\limits_{k=6}^{\infty} \frac{1}{1+k^2}$ by approximating the area under $y = \frac{1}{1+x^2}$ from 5 to infinity. See my picture. Use rectangles with base 1, which under estimate the area under the curve. The area inside the rectangles is $\sum\limits_{k=6}^{\infty} \frac{1}{1+k^2}$. The area under the curve is

$$\int_5^{\infty} \frac{1}{1+x^2} dx = \lim_{b\to\infty} \int_5^{b} \frac{1}{1+x^2} dx = \lim_{b\to\infty} \arctan x \Big|_5^b = \lim_{b\to\infty} \arctan b - \arctan 5$$

$$= \frac{\pi}{2} - \arctan 5.$$

I conclude that $\sum\limits_{k=1}^{5} \frac{1}{1+k^2}$ approximates $\sum\limits_{k=1}^{\infty} \frac{1}{1+k^2}$ with an error of at most $\frac{\pi}{2} - \arctan 5$. (I used my calculator to see that $\frac{\pi}{2} - \arctan 5 < .2$)