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Quiz – March 25, 2004

Find the sum of the series $\sum_{k=1}^{\infty} \frac{2}{(k+2)k}$. Show complete coherent work.

Answer: This is not an easy problem. Usually, the only series for which we can find the sum are geometric series or telescoping series. This series is not a geometric series. Maybe it is a telescoping series. That is, maybe $\frac{2}{(k+2)k}$ is equal to the difference of two nearby fractions. A sum of such things would telescope. Well, we could use the technique of partial fractions to see. Consider

$$\frac{2}{(k+2)k} = \frac{A}{k} + \frac{B}{k+2}.$$

Multiply through by $k(k+2)$ to see

$$2 = A(k+2) + Bk.$$

Plug $k = 0$ in to see $A = 1$. Plug $k = -2$ in to see that $B = -1$. Check that

$$\frac{1}{k} + \frac{-1}{k+2} = \frac{k+2-k}{k(k+2)} = \frac{2}{(k+2)k}.$$

The n^{th} partial sum of the series is

$$s_n = \sum_{k=1}^{n} \frac{2}{(k+2)k} = \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{-1}{k+2}\right)$$

$$= \left(\frac{1}{1} + \frac{-1}{3}\right) + \left(\frac{1}{2} + \frac{-1}{4}\right) + \left(\frac{1}{3} + \frac{-1}{5}\right) + \left(\frac{1}{4} + \frac{-1}{6}\right) + \dots$$

$$\dots + \left(\frac{1}{n-3} + \frac{-1}{n-1}\right) + \left(\frac{1}{n-2} + \frac{-1}{n}\right) + \left(\frac{1}{n-1} + \frac{-1}{n+1}\right) + \left(\frac{1}{n} + \frac{-1}{n+2}\right).$$

Cancel the $\frac{1}{3}$'s, the $\frac{1}{4}$'s, the $\frac{1}{5}$'s, the $\frac{1}{6}$'s, ... , the $\frac{1}{n-3}$'s, the $\frac{1}{n-2}$'s, the $\frac{1}{n-1}$'s, and the $\frac{1}{n}$'s. We are left with

$$s_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$

We notice that the limit of the sequence of partial sums is equal to

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}.$$

We conclude that the series converges to $\frac{3}{2}$.