Quiz – March 2, 2004

Find
\[ \int \frac{2x^2 + x - 8}{x^3 + 4x} \, dx. \]

**Answer:** Set
\[ \frac{2x^2 + x - 8}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}. \]

Multiply both sides by \( x(x^2 + 4) \) to get
\[ 2x^2 + x - 8 = A(x^2 + 4) + (Bx + C)x. \]

That is,
\[ 2x^2 + x - 8 = (A + B)x^2 + Cx + 4A. \]

Equate the corresponding coefficients to get:
\[ 2 = A + B \]
\[ 1 = C \]
\[ -8 = 4A. \]

The bottom equation says \( A = -2 \). The middle equation says \( C = 1 \). The top equation says \( B = 4 \). We check what we have so far:

\[ \frac{-2}{x} + \frac{4x + 1}{x^2 + 4} = \frac{-2(x^2 + 4) + (4x + 1)x}{x(x^2 + 4)} = \frac{2x^2 + x - 8}{x(x^2 + 4)}. \]

The original problem is
\[ \int \left( \frac{-2}{x} + \frac{4x + 1}{x^2 + 4} \right) \, dx = -2 \ln |x| + 2 \ln (x^2 + 4) + \frac{1}{2} \arctan \left( \frac{x}{2} \right) + C. \]

By the way, the derivative of \( \frac{1}{2} \arctan \left( \frac{x}{2} \right) \) is
\[ \frac{1}{2} \frac{1}{1 + \left( \frac{x}{2} \right)^2} = \frac{1}{4 \left( 1 + \frac{x^2}{4} \right)} = \frac{1}{4 + x^2}, \]
as expected.