Consider the sequence \( \{ a_n \} \) with \( a_1 = 10 \), and \( a_n = \frac{1}{2} [a_{n-1} + \frac{3}{a_{n-1}}] \) for \( n \geq 2 \). Prove that the sequence \( \{ a_n \} \) converges. Find the limit of the sequence \( \{ a_n \} \).

**Answer:** Suppose, for the time being, that the sequence converges. Let \( L = \lim_{n \to \infty} a_n \). Take the limit of both sides of \( a_n = \frac{1}{2} [a_{n-1} + \frac{3}{a_{n-1}}] \) to see that

\[
L = \frac{1}{2} [L + \frac{3}{L}].
\]

Multiply both sides by \( 2L \) to see that \( 2L^2 = L^2 + 3 \); so, \( L^2 = 3 \) and \( L \) is equal to \( \sqrt{3} \) or \( -\sqrt{3} \). All of the numbers \( a_n \) are non-negative; so \( L \) must be non-negative. We now know that, if \( L \) exists, then \( L \) must be \( \sqrt{3} \).

We still have to prove that \( L \) exists. I will show that the sequence \( \{ a_n \} \) is a decreasing sequence of Real numbers which is bounded below by \( \sqrt{3} \). The (dual of the) Completeness axiom tells us that the sequence \( \{ a_n \} \) has a limit.

I first show that \( \sqrt{3} \leq a_n \) for all \( n \). We see that \( \sqrt{3} \leq a_1 \). In general, we hope to show that

\[
\sqrt{3} \leq \frac{1}{2} [a_{n-1} + \frac{3}{a_{n-1}}].
\]

Multiply both sides by the positive number \( 2a_{n-1} \). We hope to show

\[
2\sqrt{3}a_{n-1} \leq a_{n-1}^2 + 3.
\]

We hope to show that

\[
0 \leq a_{n-1}^2 - 2\sqrt{3}a_{n-1} + 3.
\]

The right side factors as \( (a_{n-1} - \sqrt{3})^2 \), and this perfect square is non-negative. Read the calculation from the bottom up to see that \( \sqrt{3} \leq a_n \) for all \( n \). That is, We know that \( (a_{n-1} - \sqrt{3})^2 \) is a perfect square and is non-negative. Expand to get \( 0 \leq a_{n-1}^2 - 2\sqrt{3}a_{n-1} + 3 \). Add \( 2\sqrt{3}a_{n-1} \) to both sides to see that

\[
2\sqrt{3}a_{n-1} \leq a_{n-1}^2 + 3.
\]

Divide both sides by the non-negative number \( 2a_{n-1} \) to see that

\[
\sqrt{3} \leq \frac{1}{2} [a_{n-1} + \frac{3}{a_{n-1}}].
\]

The right side is equal to \( a_n \). We have shown that \( \sqrt{3} \leq a_n \) for all \( n \).
Finally, I show that $a_n \leq a_{n-1}$, for all $n \geq 2$. I will show that

$$\frac{1}{2}[a_{n-1} + \frac{3}{a_{n-1}}] \leq a_{n-1}.$$ 

Multiply by the positive number $2a_{n-1}$. We hope to show that

$$a_{n-1}^2 + 3 \leq 2a_{n-1}^2.$$ 

We hope to show that

$$0 \leq a_{n-1}^2 - 3$$

We hope to show that

$$0 \leq (a_{n-1} + \sqrt{3})(a_{n-1} - \sqrt{3}).$$

Divide by the positive number $(a_{n-1} + \sqrt{3})$. We hope to show

$$0 \leq a_{n-1} - \sqrt{3}.$$ 

Fortunately, we have already shown that every member of the sequence is at least $\sqrt{3}$. Read the calculation from the bottom to the top to see that $a_n \leq a_{n-1}$. That is, we already showed that

$$0 \leq a_{n-1} - \sqrt{3}$$

for all $n \geq 2$. Multiply both sides by the positive number $(a_{n-1} + \sqrt{3})$ to see that

$$0 \leq a_{n-1}^2 - 3$$

for all $n \geq 3$. Add $a_{n-1}^2 + \sqrt{3}$ to both sides to see that

$$a_{n-1}^2 + 3 \leq 2a_{n-1}^2.$$ 

Divide both sides by the positive number $2a_{n-1}$ to see

$$\frac{1}{2}[a_{n-1} + \frac{3}{a_{n-1}}] \leq a_{n-1}.$$ 

The left side is $a_n$. We conclude that $a_n \leq a_{n-1}$ for all $n \geq 2$. 