PRINT Your Name:____________________

Quiz – October 31, 2006

Consider the sequence $\{a_n\}$ with $a_1 = 10$, and $a_n = \frac{1}{2}[a_{n-1} + \frac{3}{a_{n-1}}]$ for $n \geq 2$. Prove that the sequence $\{a_n\}$ converges. Find the limit of the sequence $\{a_n\}$.

Answer: Suppose, for the time being, that the sequence converges. Let $L = \lim_{n\to\infty} a_n$. Take the limit of both sides of $a_n = \frac{1}{2}[a_{n-1} + \frac{3}{a_{n-1}}]$ to see that

$$L = \frac{1}{2}[L + \frac{3}{L}].$$

Multiply both sides by $2L$ to see that $2L^2 = L^2 + 3$; so, $L^2 = 3$ and L is equal to $\sqrt{3}$ or $-\sqrt{3}$. All of the numbers a_n are non-negative; so L must be non-negative. We now know that, if L exists, then L must be $\sqrt{3}$.

We still have to prove that L exists. I will show that the sequence $\{a_n\}$ is a decreasing sequence of Real numbers which is bounded below by $\sqrt{3}$. The (dual of the) Completeness axiom tells us that the sequence $\{a_n\}$ has a limit.

I first show that $\sqrt{3} \leq a_n$ for all n. We see that $\sqrt{3} \leq a_1$. In general, we hope to show that

$$\sqrt{3} \leq \frac{1}{2}[a_{n-1} + \frac{3}{a_{n-1}}].$$

Multiply both sides by the positive number $2a_{n-1}$. We hope to show

$$2\sqrt{3}a_{n-1} \leq a_{n-1}^2 + 3.$$

We hope to show that

$$0 \leq a_{n-1}^2 - 2\sqrt{3}a_{n-1} + 3.$$

The right side factors as $(a_{n-1} - \sqrt{3})^2$, and this perfect square is non-negative. Read the calculation from the bottom up to see that $\sqrt{3} \leq a_n$ for all n. That is, We know that $(a_{n-1} - \sqrt{3})^2$ is a perfect square and is non-negative. Expand to get $0 \leq a_{n-1}^2 - 2\sqrt{3}a_{n-1} + 3$. Add $2\sqrt{3}a_{n-1}$ to both sides to see that

$$2\sqrt{3}a_{n-1} \leq a_{n-1}^2 + 3.$$

Divide both sides by the non-negative number $2a_{n-1}$ to see that

$$\sqrt{3} \leq \frac{1}{2}[a_{n-1} + \frac{3}{a_{n-1}}].$$

The right side is equal to a_n. We have shown that $\sqrt{3} \leq a_n$ for all n.

Finally, I show that $a_n \leq a_{n-1}$, for all $n \geq 2$. I will show that

$$\tfrac{1}{2}[a_{n-1} + \tfrac{3}{a_{n-1}}] \leq a_{n-1}.$$

Multiply by the positive number $2a_{n-1}$. We hope to show that

$$a_{n-1}^2 + 3 \leq 2a_{n-1}^2.$$

We hope to show that

$$0 \leq a_{n-1}^2 - 3$$

We hope to show that

$$0 \leq (a_{n-1} + \sqrt{3})(a_{n-1} - \sqrt{3}).$$

Divide by the positive number $(a_{n-1} + \sqrt{3})$. We hope to show

$$0 \leq a_{n-1} - \sqrt{3}.$$

Fortunately, we have already shown that every member of the sequence is at least $\sqrt{3}$. Read the calculation from the bottom to the top to see that $a_n \leq a_{n-1}$. That is, we already showed that

$$0 \leq a_{n-1} - \sqrt{3}$$

for all $n \geq 2$. Multiply both sides by the positive number $(a_{n-1} + \sqrt{3})$ to see that

$$0 \leq a_{n-1}^2 - 3$$

for all $n \geq 3$. Add $a_{n-1}^2 + \sqrt{3}$ to both sides to see that

$$a_{n-1}^2 + 3 \leq 2a_{n-1}^2.$$

Divide both sides by the positive number $2a_{n-1}$ to see

$$\tfrac{1}{2}[a_{n-1} + \tfrac{3}{a_{n-1}}] \leq a_{n-1}.$$

The left side is a_n . We conclude that $a_n \leq a_{n-1}$ for all $n \geq 2$.