Quiz – October 29, 2004

Does the series \( \sum_{k=1}^{\infty} \frac{2}{(k+2)k} \) converge or diverge? If the series converges, find its sum.

**Answer:** If one thinks to try partial fractions on the fraction \( \frac{2}{(k+2)k} \), then one can see that this series is a telescoping series and thereby find the sum of the series. Multiply
\[
\frac{2}{(k+2)k} = \frac{A}{k+2} + \frac{B}{k}
\]
by \( (k+2)k \) to see that
\[
2 = Ak + B(k+2)
\]
or
\[
2 = (A + B)k + 2B.
\]
Equate the corresponding coefficients to see that \( B = 1 \) and \( A = -1 \). Thus, the series is
\[
\sum_{k=1}^{\infty} \left[ \frac{1}{k} - \frac{1}{k+2} \right].
\]
The \( n \)th partial sum is
\[
s_n = \sum_{k=1}^{n} \left[ \frac{1}{k} - \frac{1}{k+2} \right] = \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{n-2} - \frac{1}{n} \right) + \left( \frac{1}{n-1} - \frac{1}{n+1} \right) + \left( \frac{1}{n} - \frac{1}{n+2} \right) = \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.
\]
The sum of the series is the limit of the sequence of partial sums. So the sum of the series is
\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{3}{2}.
\]
We conclude that the series converges to \( \frac{3}{2} \).