

Quiz 7 October 6, 2010 – Section 9 – 10:10 – 11:00

Circle your answer. Show your work.

The quiz is worth 5 points.

Show that the sequence defined by $a_1 = 2$, $a_{n+1} = \frac{1}{3-a_n}$ satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

Answer. We first show that the sequence $\{a_n\}$ is decreasing. We must show that $a_{n+1} \leq a_n$. We must show that $\frac{1}{3-a_n} \leq a_n$. We must show that $-a_n + \frac{1}{3-a_n} \leq 0$. We must show that $\frac{a_n^2 - 3a_n + 1}{3-a_n} \leq 0$. The roots of the numerator are $a_n = \frac{3 \pm \sqrt{5}}{2}$. We see that the sign of the expression $\frac{a_n^2 - 3a_n + 1}{3-a_n}$ is given by:



to show that $\{a_n\}$ is decreasing, we must show that

$$(*) \quad \frac{3 - \sqrt{5}}{2} \leq a_n \leq \frac{3 + \sqrt{5}}{2}$$

for all n . (Of course, $\frac{3-\sqrt{5}}{2}$ is a little more than 0 and $\frac{3+\sqrt{5}}{2}$ is a little less than 3.) It certainly is true that $\frac{3-\sqrt{5}}{2} \leq a_1 \leq \frac{3+\sqrt{5}}{2}$. Suppose that $\frac{3-\sqrt{5}}{2} \leq a_{n-1} \leq \frac{3+\sqrt{5}}{2}$, then

$$-\frac{3 + \sqrt{5}}{2} \leq -a_{n-1} \leq -\frac{3 - \sqrt{5}}{2}$$

and

$$3 - \frac{3 + \sqrt{5}}{2} \leq 3 - a_{n-1} \leq 3 - \frac{3 - \sqrt{5}}{2}$$

We see that $3 - \frac{3+\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$ and $3 - \frac{3-\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$. We have

$$\frac{3 - \sqrt{5}}{2} \leq 3 - a_{n-1} \leq \frac{3 + \sqrt{5}}{2}$$

All of these numbers are positive; so,

$$\frac{2}{3 + \sqrt{5}} \leq \frac{1}{3 - a_{n-1}} \leq \frac{2}{3 - \sqrt{5}}$$

We see that $\frac{1}{3-a_{n-1}} = a_n$. We calculate

$$\frac{2}{3-\sqrt{5}} = \frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{2(3+\sqrt{5})}{4} = \frac{3+\sqrt{5}}{2}$$

and

$$\frac{2}{3+\sqrt{5}} = \frac{2(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = \frac{2(3-\sqrt{5})}{4} = \frac{3-\sqrt{5}}{2}.$$

If (*) holds at $n-1$, then we have shown that

$$\frac{3-\sqrt{5}}{2} \leq a_n \leq \frac{3+\sqrt{5}}{2}.$$

In other words

$$(*) \text{ holds at } n-1 \implies (*) \text{ holds at } n.$$

We also know that (*) holds at 1. It follows that (*) holds at all n . We have shown that $\{a_n\}$ is a decreasing sequence. We have $a_1 = 2$ and $\{a_n\}$ is a decreasing sequence; thus, $a_n \leq 2$ for all n .

The completeness axiom yields that $\{a_n\}$ converges. Let $L = \lim_{n \rightarrow \infty} a_n$. Take $\lim_{n \rightarrow \infty}$ of both sides of $a_{n+1} = \frac{1}{3-a_n}$ to obtain $\lim_{n \rightarrow \infty} a_{n+1} = \frac{1}{3-\lim_{n \rightarrow \infty} a_n}$ or $L = \frac{1}{3-L}$. Multiply both sides by $3-L$ to obtain $L(3-L) = 1$; so $3L-L^2 = 1$. It follows that $L = \frac{3-\sqrt{5}}{2}$ or $L = \frac{3+\sqrt{5}}{2}$. We know that $L \neq \frac{3+\sqrt{5}}{2}$ because $\frac{3+\sqrt{5}}{2} > 2 = a_1 \geq a_2 \geq a_3 \dots$. Thus, L must equal $\frac{3-\sqrt{5}}{2}$.