PRINT Your Name:____________________

Quiz 7 — February 24, 2012 – Section 7 – 11:15 – 12:05

Remove everything from your desk except a pencil or pen.

Write in complete sentences.
The quiz is worth 5 points.

Consider the sequence defined by $a_1 = 2$ and $a_{n+1} = \frac{1}{3-a_n}$.
1. Prove that $0 < a_n \le 2$ for all positive integers n.
2. Prove that $a_{n+1} \le a_n$ for all positive integers n.
3. State the Completeness Axiom and draw a conclusion about the sequence $\{a_n\}$ from the Completeness Axiom.
4. Find the limit of the sequence $\{a_n\}$.

Answer:

(1) We use the technique of Mathematical Induction. We see that $a_1 = 2$ and therefore, $0 < a_1 \le 2$. Assume **BY INDUCTION** that $0 < a_{n-1} \le 2$ for some **FIXED** n. Multiply by -1 to see $-2 \le -a_{n-1} < 0$. Add 3 to see $1 \le 3-a_{n-1} < 3$; that is $1 \le 3 - a_{n-1}$ and $3 - a_{n-1} < 3$. Divide the first inequality by the positive number $3-a_{n-1}$ to obtain $\frac{1}{3-a_{n-1}} \le 1$. Divide the second inequality by the positive number $(3 - a_{n-1})3$ to see $\frac{1}{3} < \frac{1}{3-a_{n-1}}$. Put the inequalities back together to see: $\frac{1}{3} < \frac{1}{3-a_{n-1}} \le 1$. We have shown that

$$0 < a_{n-1} \le 2 \implies \frac{1}{3} < \frac{1}{3 - a_{n-1}} \le 1.$$

Obviously, $\frac{1}{3-a_{n-1}} = a_n$, $0 < \frac{1}{3}$ and $1 \le 2$; so,

$$0 < a_{n-1} \le 2 \implies 0 < a_n \le 2.$$

We saw that $0 < a_1 \le 2$. for $n = 1$. We proved that if $0 < a_{n-1} \le 2$ for some FIXED n, then $0 < a_n \le 2$ also holds for that one FIXED n. We apply the Principle of Mathematical Induction to conclude that $0 < a_n \le 2$ for ALL positive integers n.

(2) We use the technique of Mathematical Induction. We see that $a_1 = 2$ and $a_2 = 1$; so $a_2 \le a_1$. Assume **BY INDUCTION** that $a_n \le a_{n-1}$ for some **FIXED** n. Add $-a_n - a_{n-1}$ to both sides to see $-a_{n-1} \le -a_n$. Add 3 to both sides to see: $3 - a_{n-1} \le 3 - a_n$. Both numbers are positive because part (1) shows that $a_n \le 2$ for all n. Divide both sides by the positive number $(3 - a_{n-1})(3 - a_n)$ to obtain $\frac{1}{3-a_n} \le \frac{1}{3-a_{n-1}}$ and this is $a_{n+1} \le a_n$. Thus

$$a_n \le a_{n-1} \implies a_{n+1} \le a_n.$$

We saw that $a_{n+1} \le a_n$ for $n = 1$. We proved that if $a_n \le a_{n-1}$ for some FIXED n, then $a_{n+1} \le a_n$ also holds for that one FIXED n. We apply the Principle of Mathematical Induction to conclude that $a_{n+1} \le a_n$ for ALL positive integers n.

(3) The completeness axiom says that every decreasing bounded sequence of real numbers has a limit. We showed in (1) and (2) that $\{a_n\}$ is an decreasing bounded sequence of real numbers. We conclude that $\lim\limits_{n\to\infty} a_n$ exists. Let $L = \lim\limits_{n\to\infty} a_n$.

(4) Take $\lim\limits_{n\to\infty}$ of both sides of $a_{n+1} = \frac{1}{3-a_n}$ to conclude that

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{3 - \lim\limits_{n\to\infty} a_n};$$

that is, $L = \frac{1}{3-L}$; so $L(3-L) = 1$ or $-L^2 + 3L = 1$. We use the quadratic formula to solve $0 = L^2 - 3L + 1$. We obtain $L = \frac{3\pm\sqrt{9-4}}{2}$. We know that L can not be more than 2 because every term in the sequence is less than or equal to 2. So $L \ne \frac{3+\sqrt{5}}{2}$ and hence L does equal $\frac{3-\sqrt{5}}{2}$.