

Quiz 7 — October 6, 2010 – Section 10 – 11:15 – 12:05

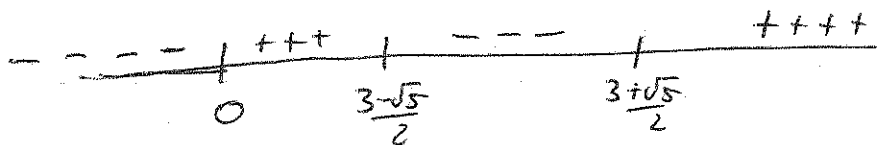
Circle your answer. Show your work.

The quiz is worth 5 points.

Show that the sequence defined by $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$ is increasing and $a_n < 3$ for all n . Deduce that the sequence is convergent and find its limit.

Answer. We need to show that $a_n \leq a_{n+1}$. We need to show that $a_n \leq 3 - \frac{1}{a_n}$.

We need to show that $a_n - 3 + \frac{1}{a_n} \leq 0$. We need to show that $\frac{a_n^2 - 3a_n + 1}{a_n} \leq 0$. The numerator has roots $a_n = \frac{3 \pm \sqrt{9-4}}{2}$. So the expression $\frac{a_n^2 - 3a_n + 1}{a_n}$ takes the following signs



We must show that

$$(*) \quad \frac{3 - \sqrt{5}}{2} \leq a_n \leq \frac{3 + \sqrt{5}}{2}$$

for all n . (Of course, $\frac{3 - \sqrt{5}}{2}$ is a little more than 0 and $\frac{3 + \sqrt{5}}{2}$ is a little less than 3.)

It certainly is true that $\frac{3 - \sqrt{5}}{2} \leq a_1 \leq \frac{3 + \sqrt{5}}{2}$. Suppose that $\frac{3 - \sqrt{5}}{2} \leq a_{n-1} \leq \frac{3 + \sqrt{5}}{2}$, then

$$\frac{2}{3 + \sqrt{5}} \leq \frac{1}{a_{n-1}} \leq \frac{2}{3 - \sqrt{5}}$$

and

$$-\frac{2}{3 - \sqrt{5}} \leq -\frac{1}{a_{n-1}} \leq -\frac{2}{3 + \sqrt{5}}$$

and

$$3 - \frac{2}{3 - \sqrt{5}} \leq 3 - \frac{1}{a_{n-1}} \leq 3 - \frac{2}{3 + \sqrt{5}}$$

But $3 - \frac{1}{a_{n-1}} = a_n$,

$$3 - \frac{2}{3 + \sqrt{5}} = 3 - \frac{2(3 - \sqrt{5})}{(3 + \sqrt{5})(3 - \sqrt{5})} = 3 - \frac{2(3 - \sqrt{5})}{4} = 3 - \frac{(3 - \sqrt{5})}{2} = \frac{(3 + \sqrt{5})}{2},$$

and

$$3 - \frac{2}{3 - \sqrt{5}} = 3 - \frac{2(3 + \sqrt{5})}{(3 - \sqrt{5})(3 + \sqrt{5})} = 3 - \frac{2(3 + \sqrt{5})}{4} = 3 - \frac{(3 + \sqrt{5})}{2} = \frac{3 - \sqrt{5}}{2}.$$

We have shown that (*) holds at $n = 1$ and if (*) holds at $n - 1$, then (*) holds at n . We conclude that (*) holds at all n .

In particular, we have shown that the sequence $\{a_n\}$ is increasing and that $a_n < 3$ for all n .

The completeness axiom yields that $\{a_n\}$ converges. Let $L = \lim_{n \rightarrow \infty} a_n$. Take $\lim_{n \rightarrow \infty}$ of both sides of $a_{n+1} = 3 - \frac{1}{a_n}$ to obtain $\lim_{n \rightarrow \infty} a_{n+1} = 3 - \frac{1}{\lim_{n \rightarrow \infty} a_n}$ or $L = 3 - \frac{1}{L}$. Multiply both sides by L to obtain $L^2 = 3L - 1$; so $L = \frac{3 - \sqrt{5}}{2}$ or $L = \frac{3 + \sqrt{5}}{2}$. We know that $L \neq \frac{3 - \sqrt{5}}{2}$ because $\frac{3 - \sqrt{5}}{2} < 1 = a_1 \leq a_2 \leq a_3 \dots$. Thus, L must equal $\frac{3 + \sqrt{5}}{2}$.