Find $\int \frac{dx}{x^2\sqrt{9x^2-4}}$. Check your answer.

**Answer:** Let $3x = 2\sec \theta$. It follows that $3dx = 2\sec \theta \tan \theta \, d\theta$,

$$\sqrt{9x^2 - 4} = \sqrt{4\sec^2 \theta - 4} = 2 \tan \theta,$$

and the original integral is equal to

$$\int \frac{(2/3) \sec \theta \tan \theta \, d\theta}{(4/9) \sec^2 \theta (2) \tan \theta} = \frac{3}{4} \int \frac{d\theta}{\sec \theta} = \frac{3}{4} \int \cos \theta \, d\theta = \frac{3}{4} \sin \theta + C.$$

Consider a triangle with hypotenuse $3x$, adjacent $2$, and opposite $\sqrt{9x^2 - 4}$. We see that the answer is

$$\frac{3}{4} \frac{\sqrt{9x^2 - 4}}{3x} + C = \frac{\sqrt{9x^2 - 4}}{4x} + C.$$

**Check:** The derivative of the proposed answer is

$$\frac{4x \cdot \frac{18x}{2\sqrt{9x^2-4}} - 4\sqrt{9x^2-4}}{16x^2} = \frac{\frac{9x^2}{\sqrt{9x^2-4}} - \sqrt{9x^2-4}}{4x^2} = \frac{9x^2 - (9x^2 - 4)}{4x^2 \sqrt{9x^2 - 4}} = \frac{4}{4x^2 \sqrt{9x^2 - 4}} = \frac{\sqrt{9x^2 - 4}}{4x}.$$