Quizzes 10 and 11 — April 6, 2011 – Section 4 – 9:05-9:55 recitation.

Each question is worth 5 points.

1. Does the series $\sum\limits_{n=2}^{\infty} \frac{1+4^n}{1+3^n}$ converge? **Justify your answer very thoroughly. Use complete sentences.**

2. Does the series $\sum\limits_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converge? **Justify your answer very thoroughly. Use complete sentences.**

Answer. 1. We compare $\sum\limits_{n=2}^{\infty} \frac{1+4^n}{1+3^n}$ to $\sum\limits_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$. The series $\sum\limits_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$ is the geometric series with ratio $r = \frac{4}{3} > 1$; thus, $\sum\limits_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$ diverges. We use the Limit Comparison Test. We see that

$$\lim_{n\to\infty} \frac{\frac{1+4^n}{1+3^n}}{\left(\frac{4}{3}\right)^n} = \lim_{n\to\infty} \frac{1+4^n}{1+3^n}\left(\frac{3}{4}\right)^n = \lim_{n\to\infty} \frac{\frac{1}{3^n}+\frac{4^n}{3^n}}{\frac{1}{3^n}+1}\left(\frac{3}{4}\right)^n = \lim_{n\to\infty} \frac{\frac{1}{4^n}+1}{\frac{1}{3^n}+1} = 1.$$

The limit 1 is a number. This number is not zero or infinity; so the Limit Comparison Test guarantees that $\sum\limits_{n=2}^{\infty} \frac{1+4^n}{1+3^n}$ and $\sum\limits_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$ both converge or both diverge. We have seen that $\sum\limits_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$ diverges. We conclude that $\sum\limits_{n=2}^{\infty} \frac{1+4^n}{1+3^n}$ also diverges.

2. We apply the integral test. We see that $f(x) = \frac{1}{x\sqrt{\ln x}}$ is a positive decreasing function. The integral test guarantees that the series $\sum\limits_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ and the integral $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx$ both converge or both diverge. We compute the integral. Let $u = \ln x$. It follows that $du = \frac{1}{x}dx$. When $x = 2$, we have $u = \ln 2$. When x goes to infinity, then u also goes to infinity. We have

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx = \int_{\ln 2}^{\infty} u^{-1/2}du = \lim_{b\to\infty} 2\sqrt{u}\Big|_{\ln 2}^{b} = \lim_{b\to\infty} (2\sqrt{b} - 2\sqrt{\ln 2}) = \infty.$$

The integral diverges. Thus, the series $\sum\limits_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ also diverges.