Math 142, Exam 3, Fall 1998, Problems 4, 5, 6, and 7

4. Find $\int \frac{5x^2 - 2x + 2}{x^3 + x} dx$.

Write

$$\frac{5x^2 - 2x + 2}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Multiply both sides by $x^3 + x$ **to get**

$$\begin{aligned} 5x^2 - 2x + 2 =& A(x^2 + 1) + (Bx + C)x \\ 5x^2 - 2x + 2 =& (A + B)x^2 + Cx + A. \end{aligned}$$

Equate the corresponding coefficients:

$$\begin{aligned} 5 &= A + B \\ -2 &= C \\ 2 &= A. \end{aligned}$$

We see that $B = 3$ **and**

$$\int \frac{5x^2 - 2x + 2}{x^3 + x} dx = \int \frac{2}{x} + \frac{3x - 2}{x^2 + 1} dx = \boxed{2 \ln |x| + \frac{3}{2} \ln(x^2 + 1) - 2 \arctan x + C.}$$

5. Find $\int \sin^2 x \cos^3 x dx$.

Let $u = \sin x$ **. In this case** $du = \cos x \, dx$ **and**

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x(1 - \sin^2 x) \cos x dx = \int (u^2 - u^4) du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}.$$

6. (10 points for each part)
   (a) Find the Taylor polynomial of degree three, $P_3(x)$, for $f(x) = e^{-x}$ about $a = 0$.
   (b) Find an upper bound for the difference between $f(x)$ and $P_3(x)$ when $|x| \le \frac{1}{10}$.

We see that

$$\begin{array}{ll} f(x) = e^{-x} & f(0) = 1 \\ f'(x) = -e^{-x} & f'(0) = -1 \\ f''(x) = e^{-x} & f''(0) = 1 \\ f'''(x) = -e^{-x} & f'''(0) = -1 \end{array}$$

So,

$$\boxed{P_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!}.}$$

We know that $f^{(4)}(x) = e^{-x}$ **. Therefore, the difference**

$$|f(x) - P_3(x)| = |R_3(X)| = \left| \frac{f^{(4)}(c)x^4}{4!} \right| = \frac{|x|^4}{4!e^c}$$

for some c **between** x **and** 0 **. We know that** $-\frac{1}{10} \leq c \leq \frac{1}{10}$ **. It follows that**

$$e^{-\frac{1}{10}} \leq e^c \leq e^{\frac{1}{10}}$$

and

$$\frac{1}{e^{\frac{1}{10}}} \leq \frac{1}{e^c} \leq \frac{1}{e^{-\frac{1}{10}}} = e^{\frac{1}{10}}.$$

Thus,

$$|f(x) - P_3(x)| = \frac{|x|^4}{4!e^c} \leq \frac{(\frac{1}{10})^4 e^{\frac{1}{10}}}{4!} = \frac{e^{\frac{1}{10}}}{4! \cdot 10^4}.$$

We conclude that

$$\boxed{|f(x) - P_3(x)| \leq \frac{e^{\frac{1}{10}}}{4! \cdot 10^4}.}$$

7. Find $\lim_{x \to \frac{\pi}{4}} \frac{\sin x}{x}$.

It is clear that

$$\lim_{x \to \frac{\pi}{4}} \frac{\sin x}{x} = \boxed{\frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}}.}$$