Exam 1, Math 142, Fall 1998, Problems 3, 4, 9, and 10

3. Find the area of the region bounded by $y = e^x$, and the line through $(0,1)$ and $(1,e)$.
The equation of the line is $y - 1 = (e-1)x$, or $y = (e-1)x + 1$. For our region, the line has a larger y-coordinate than the exponential curve. So, the area is

$$\int_0^1 (e-1)x + 1 - e^x \, dx = (e-1)\frac{x^2}{2} + x - e^x \Big|_0^1 = \frac{e-1}{2} + 1 - e - (-e^0)$$
$$= \frac{e}{2} - \frac{1}{2} + 1 - e + 1 = \boxed{\frac{3}{2} - \frac{e}{2}}.$$

4. Simplify $\cos\left[\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right]$.
We have

$$\cos\left[\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right]$$
$$= \cos\left(\cos^{-1}\left(\frac{4}{5}\right)\right)\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right) - \sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)\sin\left(\sin^{-1}\left(\frac{12}{13}\right)\right).$$

Think about the right triangle whose sides are $3-4-5$ to see that $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \frac{3}{5}$. Think about the right triangle whose sides are $5-12-13$ to see that $\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right) = \frac{5}{13}$. Thus,

$$\cos\left[\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right]$$
$$= \left(\frac{4}{5}\right)\frac{5}{13} - \frac{3}{5}\frac{12}{13} = \frac{20-36}{65} = \boxed{\frac{-16}{65}.}$$

9. Find $\int \frac{1}{x \ln x} dx$.
Let $u = \ln x$. It follows that $du = \frac{1}{x} dx$. Thus,

$$\int \frac{1}{x \ln x} dx = \int \frac{du}{u} = \ln|u| + c = \boxed{\ln|\ln x| + C}.$$

10. Find $\int (\frac{1}{e^{2x}} + \frac{1}{2x-1}) dx$.
We see that

$$\int \left(\frac{1}{e^{2x}} + \frac{1}{2x-1}\right) dx = \int \left(e^{-2x} + \frac{1}{2x-1}\right) dx = \boxed{\frac{e^{-2x}}{-2} + \frac{1}{2}\ln|2x-1| + C}.$$