Math 142, Fall 2004, Exam 1, Solutions

PRINT Your Name:____________________________________

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office door tomorrow morning, you may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 4:00 PM today.

1. **Find** $\int e^{5x+9}dx$ **. Check your answer.**

Let $u = 5x+9$. It follows that $du = 5dx$ and the integral is

$$\tfrac{1}{5}\int e^u du = \tfrac{1}{5}e^u + C = \boxed{\tfrac{1}{5}e^{5x+9}+C}.$$

CHECK: The derivative of the proposed answer is $\frac{1}{5}e^{5x+9}5$.✓

2. **Find** $\int \frac{1}{x\sqrt{\ln x}}dx$ **. Check your answer.**

Let $u = \ln x$. It follows that $du = \frac{dx}{x}$ and the integral is

$$\int u^{-1/2}du = 2u^{1/2} + C = \boxed{2\sqrt{\ln x}+C}.$$

CHECK: The derivative of the proposed answer is $2\frac{1}{2}(\ln x)^{-1/2}(\frac{1}{x})$. ✓

3. **If** $y = e^{\sin(2x^2+5x)}$ **, then find** $\frac{dy}{dx}$ **.**

We see that

$$\frac{dy}{dx} = \boxed{(4x+5)\cos(2x^2+5x)e^{\sin(2x^2+5x)}}.$$

4. **If** $y = x^2\ln(2x^2+9x)$ **, then find** $\frac{dy}{dx}$ **.**

We see that

$$\frac{dy}{dx} = \boxed{x^2\frac{4x+9}{2x^2+9x} + 2x\ln(2x^2+9x)}.$$

5. **If** $y = x^x$ **, then find** $\frac{dy}{dx}$ **.**

Apply ln to both sides to obtain: $\ln y = x\ln x$. Now, apply $\frac{d}{dx}$ to both sides:

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x.$$

Multiply both sides by y , which is equal to x^x , to see $\boxed{\frac{dy}{dx} = x^x(1+\ln x)}$.

6. **Solve** $\log_2 x = 1 + \frac{1}{2}\log_2(2x)$ **. Check your answer.**

Raise 2 to each side:

$$2^{\log_2 x} = 2^{1+\frac{1}{2}\log_2(2x)}.$$

This is the same as $x = 2\sqrt{2x}$. Square both sides to obtain $x^2 = 4(2x)$, or $x^2 - 8x = 0$, or $x(x-8) = 0$. So, $x = 8$ or $x = 0$. We see that $x = 0$ is not in the domain of $\log_2 x$; therefore, $\boxed{x = 8}$.

CHECK: Plug $x = 8$ into $\log_2 x = 1 + \frac{1}{2}\log_2(2x)$. We hope that $\log_2 8 = 1 + \frac{1}{2}\log_2(16)$. We see that 3 is equal to $1 + \frac{1}{2}(4)$. ✓

7. **Find the area bounded by** $y = e^{2x}$ **, the** x**-axis,** $x = 0$**, and** $x = 1$**. Sketch a picture.**

My picture is on another page. The area is equal to

$$\int_0^1 e^{2x}dx = \tfrac{1}{2}e^{2x}|_0^1 = \boxed{\tfrac{1}{2}(e^2 - 1)}.$$

8. **Find the volume of the solid which is obtained by revolving the region bounded by** $y = e^{3x}$ **, the** x**-axis,** $x = 0$**, and** $x = 1$**, about the** x**-axis. Sketch a picture.**

My picture is on another page. Spin each rectangle, get a disk of volume $\pi r^2 t$, where the radius r is equal to the y-coordinate at the top of the rectangle minus the y-coordinate at the bottom of the rectangle, all written with respect to x . In other words, $r = e^{3x} - 0$. The thickness t is dx . The volume is

$$\pi\int_0^1 (e^{3x})^2dx = \pi\int_0^1 (e^{6x})dx = \pi\tfrac{1}{6}e^{6x}|_0^1 = \boxed{\tfrac{\pi}{6}(e^6 - 1)}.$$

9. **Let** $f(x) = \frac{x+3}{x-2}$ **for** $x \neq 2$**. Find** $f^{-1}(x)$**. What is the domain of** $f^{-1}(x)$**? Verify that** $f(f^{-1}(x)) = x$ **for all** x **in the domain of** $f^{-1}(x)$**.**

Let $y = f^{-1}(x)$. So, $f(y) = x$. In other words, y is in the domain of f (so $y \neq 2$) and $\frac{y+3}{y-2} = x$. We want to find y , so we multiply both sides by $y - 2$ to get: $y + 3 = x(y-2)$. This expression is linear in y , so we get every term with y on one side and every term without y on the other side: $2x + 3 = xy - y$, or $2x + 3 = (x-1)y$, or $y = \frac{2x+3}{x-1}$. It follows that

$$\boxed{f^{-1}(x) = \frac{2x+3}{x-1} \text{ for } x \neq 1.}$$

Take $x \neq 1$. Observe that

$$f(f^{-1}(x)) = f\left(\frac{2x+3}{x-1}\right) = \frac{\frac{2x+3}{x-1} + 3}{\frac{2x+3}{x-1} - 2}.$$

Multiply top and bottom by $x - 1$ to get:

$$\frac{2x+3+3(x-1)}{2x+3-2(x-1)} = \frac{5x}{5} = x. \checkmark$$

10. **Let $f(x) = x^2 \ln x$. What is the domain of $f(x)$? Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of $y = f(x)$. Graph $y = f(x)$.**

$\boxed{\text{The domain of } f(x) \text{ is all POSITIVE } x.}$ We see that

$$f'(x) = x^2\frac{1}{x} + 2x\ln x = x(1 + 2\ln x).$$

So, $f'(x) = 0$ for $x = e^{-1/2}$, $f'(x)$ is positive for $e^{-1/2} < x$ and $f'(x) < 0$ for $0 < x < e^{-1/2}$. In other words,

$$\boxed{\begin{array}{c} (\frac{1}{\sqrt{e}}, \frac{-1}{2e}) \text{ is the local minimum,} \\ \text{the graph does not have a local maximum,} \\ \text{the graph is decreasing for } 0 < x < \frac{1}{\sqrt{e}}, \text{ and} \\ \text{the graph is increasing for } \frac{1}{\sqrt{e}} < x. \end{array}}$$

We see that

$$f''(x) = x\frac{2}{x} + 1 + 2\ln x = 3 + 2\ln x.$$

Thus, $f''(x)$ is zero for $x = e^{-3/2}$, $f''(x)$ is positive for $e^{-3/2} < x$, and $f''(x)$ is negative for $0 < x < e^{-3/2}$. We conclude that

$$\boxed{\begin{array}{c} (\frac{1}{(\sqrt{e})^3}, \frac{-3}{2e^3}) \text{ is the point of inflection,} \\ \text{the graph is concave down for } 0 < x < \frac{1}{(\sqrt{e})^3}, \text{ and} \\ \text{the graph is concave up for } \frac{1}{(\sqrt{e})^3} < x. \end{array}}$$

In Chapter 9, you will learn L'hopital's rule which will allow you to calculate $\lim_{x \to 0^+} x^2 \ln x = 0$. I will include this on my graph, you won't know this ahead of time. But you do know that when x is near zero and above zero, then $f(x)$ is negative, decreasing, and concave down. My picture is on another page.