Math 142, Fall 2004, Exam 1, Solutions

PRINT Your Name:

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. [CIRCLE] your answer. NO CALCULATORS! CHECK your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

If you would like, I will leave your exam outside my office door tomorrow morning, you may pick it up any time between then and the next class. Let me know if you are interested.

I will post the solutions on my website at about 4:00 PM today.

1. Find \( \int e^{5x+9} \, dx \). Check your answer.

Let \( u = 5x + 9 \). It follows that \( du = 5 \, dx \) and the integral is

\[
\frac{1}{5} \int e^{u} \, du = \frac{1}{5} e^{u} + C = \frac{1}{5} e^{5x+9} + C.
\]

CHECK: The derivative of the proposed answer is \( \frac{1}{5} e^{5x+9} \). ✓

2. Find \( \int \frac{1}{x \sqrt{\ln x}} \, dx \). Check your answer.

Let \( u = \ln x \). It follows that \( du = \frac{dx}{x} \) and the integral is

\[
\int u^{-1/2} \, du = 2u^{1/2} + C = 2\sqrt{\ln x} + C.
\]

CHECK: The derivative of the proposed answer is \( 2\frac{1}{2} (\ln x)^{-1/2}(\frac{1}{x}) \). ✓

3. If \( y = e^{\sin(2x^2 + 5x)} \), then find \( \frac{dy}{dx} \).

We see that

\[
\frac{dy}{dx} = (4x + 5) \cos(2x^2 + 5x) e^{\sin(2x^2 + 5x)}.
\]

4. If \( y = x^2 \ln(2x^2 + 9x) \), then find \( \frac{dy}{dx} \).

We see that

\[
\frac{dy}{dx} = \frac{x^2 \cdot 4x + 9}{2x^2 + 9x} + 2x \ln(2x^2 + 9x).
\]

5. If \( y = x^x \), then find \( \frac{dy}{dx} \).

Apply \( \ln \) to both sides to obtain: \( \ln y = x \ln x \). Now, apply \( \frac{d}{dx} \) to both sides:

\[
\frac{1}{y} \frac{dy}{dx} = x \frac{1}{x} + \ln x.
\]

Multiply both sides by \( y \), which is equal to \( x^x \), to see

\[
\frac{dy}{dx} = x^x (1 + \ln x).
\]
6. Solve $\log_2 x = 1 + \frac{1}{2} \log_2 (2x)$. Check your answer.

Raise 2 to each side:

$2^{\log_2 x} = 2^{1+\frac{1}{2} \log_2 (2x)}$.

This is the same as $x = 2\sqrt{2x}$. Square both sides to obtain $x^2 = 4(2x)$, or $x^2 - 8x = 0$, or $x(x - 8) = 0$. So, $x = 8$ or $x = 0$. We see that $x = 0$ is not in the domain of $\log_2 x$; therefore, $x = 8$.

CHECK: Plug $x = 8$ into $\log_2 x = 1 + \frac{1}{2} \log_2 (2x)$. We hope that $\log_2 8 = 1 + \frac{1}{2} \log_2 (16)$. We see that 3 is equal to $1 + \frac{1}{2}(4)$.

7. Find the area bounded by $y = e^{2x}$, the $x$-axis, $x = 0$, and $x = 1$.

Sketch a picture.

My picture is on another page. The area is equal to

$$\int_0^1 e^{2x} dx = \frac{1}{2} e^{2x} \bigg|_0^1 = \frac{1}{2} (e^2 - 1)$$

8. Find the volume of the solid which is obtained by revolving the region bounded by $y = e^{3x}$, the $x$-axis, $x = 0$, and $x = 1$, about the $x$-axis.

Sketch a picture.

My picture is on another page. Spin each rectangle, get a disk of volume $\pi r^2 t$, where the radius $r$ is equal to the $y$-coordinate at the top of the rectangle minus the $y$-coordinate at the bottom of the rectangle, all written with respect to $x$. In other words, $r = e^{3x} - 0$. The thickness $t$ is $dx$. The volume is

$$\pi \int_0^1 (e^{3x})^2 dx = \pi \int_0^1 (e^{6x}) dx = \pi \frac{1}{6} e^{6x} \bigg|_0^1 = \frac{\pi}{6} (e^6 - 1)$$

9. Let $f(x) = \frac{x+3}{x-2}$ for $x \neq 2$. Find $f^{-1}(x)$. What is the domain of $f^{-1}(x)$?

Verify that $f(f^{-1}(x)) = x$ for all $x$ in the domain of $f^{-1}(x)$.

Let $y = f^{-1}(x)$. So, $f(y) = x$. In other words, $y$ is in the domain of $f$ (so $y \neq 2$) and $\frac{y+3}{y-2} = x$. We want to find $y$, so we multiply both sides by $y - 2$ to get: $y + 3 = x(y - 2)$. This expression is linear in $y$, so we get every term with $y$ on one side and every term without $y$ on the other side: $2x + 3 = xy - y$, or $2x + 3 = (x - 1)y$, or $y = \frac{2x+3}{x-1}$. It follows that

$$f^{-1}(x) = \frac{2x + 3}{x - 1} \text{ for } x \neq 1.$$ 

Take $x \neq 1$. Observe that

$$f(f^{-1}(x)) = f \left( \frac{2x + 3}{x - 1} \right) = \frac{2x+3}{x-1} + \frac{3}{\frac{2x+3}{x-1} - 2}.$$ 

Multiply top and bottom by $x - 1$ to get:

$$\frac{2x + 3}{2x + 3 - 2(x - 1)} = \frac{5x}{5} = x.$$

$\checkmark$
10. Let \( f(x) = x^2 \ln x \). What is the domain of \( f(x) \)? Where is \( f(x) \) increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of \( y = f(x) \). Graph \( y = f(x) \).

The domain of \( f(x) \) is all POSITIVE \( x \). We see that

\[
f'(x) = x^2 \frac{1}{x} + 2x \ln x = x(1 + 2 \ln x).
\]

So, \( f'(x) = 0 \) for \( x = e^{-1/2} \), \( f'(x) \) is positive for \( e^{-1/2} < x \) and \( f'(x) < 0 \) for \( 0 < x < e^{-1/2} \). In other words,

\[
\left( \frac{1}{\sqrt{e}}, \frac{-1}{2e} \right) \text{ is the local minimum,}
\]

\[
\text{the graph does not have a local maximum,}
\]

\[
\text{the graph is decreasing for } 0 < x < \frac{1}{\sqrt{e}}, \text{ and}
\]

\[
\text{the graph is increasing for } \frac{1}{\sqrt{e}} < x.
\]

We see that

\[
f''(x) = x \frac{2}{x} + 1 + 2 \ln x = 3 + 2 \ln x.
\]

Thus, \( f''(x) \) is zero for \( x = e^{-3/2} \), \( f''(x) \) is positive for \( e^{-3/2} < x \), and \( f''(x) \) is negative for \( 0 < x < e^{-3/2} \). We conclude that

\[
\left( \frac{1}{(\sqrt{e})^3}, \frac{-3}{2e} \right) \text{ is the point of inflection,}
\]

\[
\text{the graph is concave down for } 0 < x < \frac{1}{(\sqrt{e})^3}, \text{ and}
\]

\[
\text{the graph is concave up for } \frac{1}{(\sqrt{e})^3} < x.
\]

In Chapter 9, you will learn L’hopital’s rule which will allow you to calculate \( \lim_{x \to 0^+} x^2 \ln x = 0 \). I will include this on my graph, you won’t know this ahead of time. But you do know that when \( x \) is near zero and above zero, then \( f(x) \) is negative, decreasing, and concave down. My picture is on another page.