# Math 142, Final Exam, Fall 2002, SOLUTIONS

Name ________________________________

There are 16 problems on 8 pages. Problems 1 through 6 are worth 10 points each. Problems 7 through 16 are worth 9 points each.

✓ **Check your answer**, whenever possible.

$\boxed{CIRCLE}$ your answer. **NO CALCULATORS!**

I will post an answer key on my web site shortly after the exam is finished. If I know your e-address, I will e-mail your grade to you as soon as it is ready; otherwise, get your grade from VIP.

1. **Find** $\int x \ln x dx$ **.**

Use integration by parts. Let $u = \ln x$ and $dv = xdx$. It follows that $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$. The given integral is $\frac{x^2}{2}\ln x - \frac{1}{2}\int xdx = \boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C}$. Check. The derivative of the proposed answer is

$$\frac{x^2}{2}\left(\frac{1}{x}\right) + x\ln x - \frac{x}{2}.\checkmark$$

2. **Find** $\int \frac{\ln x}{x} dx$ **.**

Let $u = \ln x$. It follows that $du = \frac{dx}{x}$, and the answer is $\boxed{\frac{(\ln x)^2}{2} + C}$. Check the derivative of the proposed answer is $\frac{2\ln x}{2x}$ ✓.

3. **Find** $\int_2^5 \frac{dx}{(x-3)^4}$ **.**

The function $f(x) = \frac{1}{(x-3)^4}$ goes to infinity as $x$ goes to $3$. This integral is improper. The integral is equal to

$$\lim_{b\to 3^-} \left.\frac{-1}{3(x-3)^3}\right|_2^b + \lim_{a\to 3^+} \left.\frac{-1}{3(x-3)^3}\right|_a^5$$

$$= \lim_{b\to 3^-} \frac{-1}{3(b-3)^3} + \frac{1}{3(2-3)^3} + \lim_{a\to 3^+} \frac{-1}{3(2)^3} + \frac{1}{3(a-3)^3}$$

$$= +\infty - \frac{1}{3} - \frac{1}{24} + \infty.$$

$$\boxed{\text{The integral diverges to } +\infty}.$$

4. **Find** $\int \sin^2 x \cos^3 x dx$ **.**

Save one $\cos x$ . Convert the remaining $\cos^2 x$ to $1-\sin^2 x$ . The integral is equal to

$$\int \sin^2 x(1-\sin^2 x)\cos x dx.$$

Let $u = \sin x$ . So, $du = \cos x$ and the integral is

$$\int u^2(1-u^2)du = \int (u^2 - u^4)du = \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}.$$

Check. The derivative of the proposed answer is

$$\sin^2 x \cos x - \sin^4 x \cos x = \sin^2 x \cos x (1-\sin^2 x)\checkmark.$$

5. **Approximate** $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ **, with an error of at most** $\frac{1}{100}$ **.**
**Explain what you are doing!**

The series alternates. The absolute value of the terms is decreasing. The terms go to zero. I may apply the Alternating Series Test to see that

$$\boxed{\begin{array}{c}\sum\limits_{n=1}^{9} \frac{(-1)^{n+1}}{n^2} \text{ approximates } \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \\ \text{with an error of at most } \frac{1}{100}.\end{array}}$$

6. **Approximate** $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ **, with an error of at most** $\frac{1}{100}$ **. Explain what you are doing!**

The difference between $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum\limits_{n=1}^{M} \frac{1}{n^3}$ is exactly equal to $\sum\limits_{n=M+1}^{\infty} \frac{1}{n^3}$ and this difference is less than

$$\int\limits_{M}^{\infty} \frac{1}{x^3} dx.$$

Draw a picture of some boxes which underestimate the area under $f(x) = 1/x^3$ from $x = M$ to $x = \infty$. The area inside the boxes is $\sum\limits_{n=M+1}^{\infty} \frac{1}{n^3}$. The area under the curve is $\int\limits_{M}^{\infty} \frac{1}{x^3} dx$. (My picture is on a separate page.) At any rate,

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{M} \frac{1}{n^3} \right| \le \lim_{b\to\infty} \left. \frac{-1}{2x^2} \right|_M^b = \lim_{b\to\infty} \frac{-1}{2b^2} + \frac{1}{2M^2} = \frac{1}{2M^2}.$$

If $M = 8$, then $\frac{1}{2M^2} < 1/100$. I conclude that

$$\boxed{\begin{array}{c} \sum\limits_{n=1}^{8} \frac{1}{n^3} \text{ approximates } \sum\limits_{n=1}^{\infty} \frac{1}{n^3} \\ \text{with an error of at most } \frac{1}{100}. \end{array}}$$

7. **Approximate** $e^{\frac{1}{10}}$ **, with an error of at most** $\frac{1}{100}$ **. Explain what you are doing!**

We know that if $f(x) = e^x$, then $f(x) = P_n(x) + R_n(x)$, where $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$ for some $z$ between $0$ and $x$. We care about $x = \frac{1}{10}$. Our $f^{(n+1)}(x) = e^x$. It follows that

$$\left| R_n\left(\frac{1}{10}\right) \right| = \frac{e^z}{(n+1)! 10^{n+1}}$$

for some $z$ with $0 \le z \le \frac{1}{10}$. Thus, $e^z \le e^{\frac{1}{10}} < 2$. So,

$$\left| R_n\left(\frac{1}{10}\right) \right| \le \frac{2}{(n+1)!10^{n+1}}.$$

If $n = 1$, then $\frac{2}{(n+1)!10^{n+1}} = \frac{1}{100}$. It follows that

$$\left| e^{\frac{1}{10}} - P_1\left(\frac{1}{10}\right) \right| = \left| R_1\left(\frac{1}{10}\right) \right| \le \frac{1}{100}.$$

We know that $P_1(x) = 1 + x$. We conclude that

$$\boxed{\begin{array}{c}\text{1.1 approximates } e^{\frac{1}{10}} \\ \text{with an error of at most } \frac{1}{100}.\end{array}}$$

8. **Find the limit of the sequence whose $n^{\text{th}}$ term is $a_n = \left(\frac{n-2}{n}\right)^{3n}$.**

We know that $\lim\limits_{n\to\infty}(1 + \frac{r}{n})^n = e^r$. Our problem is

$$\left(\lim_{n\to\infty}\left(1 + \frac{-2}{n}\right)^n\right)^3 = \left(e^{-2}\right)^3 = \boxed{e^{-6}}.$$

9. **Find** $\displaystyle\int \frac{3x^3 - x^2 + x}{(x^2+1)^2}dx$**.**

We solve

$$\frac{3x^3 - x^2 + x}{(x^2+1)^2} = \frac{A + Bx}{(x^2+1)^2} + \frac{C + Dx}{(x^2+1)}.$$

Multiply by $(x^2+1)^2$ to get

$$3x^3 - x^2 + x = A + Bx + (C + Dx)(x^2 + 1)$$

$$3x^3 - x^2 + x = Dx^3 + Cx^2 + (B + D)x + (A + C).$$

Thus, $D = 3$, $C = -1$, $B = -2$, and $A = 1$. We check this much:

$$\frac{1-2x}{(1+x^2)^2} + \frac{-1+3x}{1+x^2} = \frac{1-2x+(-1+3x)(1+x^2)}{(1+x^2)^2}$$

$$= \frac{x - x^2 + 3x^3}{(1+x^2)^2}.$$

Now we integrate

$$\int \frac{1-2x}{(1+x^2)^2} + \frac{-1+3x}{1+x^2} dx$$

$$= \int \frac{1}{(1+x^2)^2} dx + \frac{1}{1+x^2} - \arctan x + \frac{3}{2}\ln(1+x^2) + C.$$

For the remaining integral, we let $x = \tan\theta$. It follows that $1 + x^2 = \sec^2\theta$ and $dx = \sec^2\theta d\theta$. So,

$$\int \frac{1}{(1+x^2)^2} dx = \int \frac{\sec^2\theta d\theta}{\sec^4\theta} = \int \cos^2\theta d\theta = \frac{1}{2}\int (1+\cos 2\theta) d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{1}{2}\arctan x + \frac{\sin\theta\cos\theta}{2} + C$$

$$= \frac{1}{2}\arctan x + \frac{x}{2(x^2+1)} + C.$$

(Draw the triangle with $\tan\theta = x$ (so, the opposite is $x$ and the adjacent is $1$) to see that $\sin\theta = \frac{x}{\sqrt{x^2+1}}$ and $\cos\theta = \frac{1}{\sqrt{x^2+1}}$.) At any rate, the answer is

$$\frac{1}{2}\arctan x + \frac{x}{2(x^2+1)} + \frac{1}{1+x^2} - \arctan x + \frac{3}{2}\ln(1+x^2) + C$$

$$= \boxed{\frac{1+x/2}{1+x^2} - \frac{1}{2}\arctan x + \frac{3}{2}\ln(1+x^2) + C.}$$

Check. The derivative of the proposed answer is

$$\frac{(1+x^2)(1/2)-2x(1+x/2)}{(1+x^2)^2}-\frac{1/2}{1+x^2}+\frac{3x}{1+x^2}$$

$$=\frac{(1/2)+(1/2)x^2-2x-x^2-(1/2)-(1/2)x^2+3x+3x^3}{(1+x^2)^2}$$

$$=\frac{x-x^2+3x^3}{(1+x^2)^2}.\checkmark$$

10. **Does the series** $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$ **converge? Justify your answer.**

Use the ratio test. Let

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}}=\lim_{n\to\infty}\frac{n+1}{e^{n+1}}\frac{e^n}{n}=\lim_{n\to\infty}\frac{n+1}{ne}=\frac{1}{e}.$$

Since $\rho<1$, we conclude that

$$\boxed{\sum_{n=1}^{\infty}\frac{n}{e^n}\text{ converges.}}$$

11. **Find** $\displaystyle\lim_{x\to 0}\frac{\cos x^3-1+\frac{x^6}{2}}{x^{12}}$ **.**

We know that

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\ .$$

It follows that

$$\cos x^3=1-\frac{x^6}{2!}+\frac{x^{12}}{4!}-\frac{x^{18}}{6!}+\dots\ .$$

It follows further that

$$\lim_{x\to 0} \frac{\cos x^3 - 1 + \frac{x^6}{2}}{x^{12}} = \lim_{x\to 0} \frac{\frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots}{x^{12}} = \lim_{x\to 0} \frac{1}{4!} - \frac{x^6}{6!} + \dots$$

$$= \boxed{\frac{1}{24}}.$$

12. **Find** $\int \sqrt{1+x^2}dx$ **.**

Let $x = \tan\theta$. It follows that $dx = \sec^2\theta$ and $1 + x^2 = \sec^2\theta$. The integral is equal to

$$\int \sec^3\theta d\theta.$$

Use integration by parts. Let $u = \sec\theta$ and $dv = \sec^2\theta d\theta$. It follows that $du = \sec\theta\tan\theta d\theta$ and $v = \tan\theta$. We see that

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta.$$

Thus,

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta.$$

Add $\int \sec^3\theta d\theta$ to both sides of the equation to see that

$$2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta d\theta.$$

It follows that

$$\int \sec^3\theta d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C.$$

The answer to the original problem is

$$\boxed{\frac{1}{2}\left(x\sqrt{1+x^2}+\ln\left|\sqrt{1+x^2}+x\right|\right)+C.}$$

Check. The derivative of the proposed answer is

$$\frac{1}{2}\left(\frac{x^2}{\sqrt{1+x^2}}+\sqrt{1+x^2}+\frac{\frac{x}{\sqrt{1+x^2}}+1}{\sqrt{1+x^2}+x}\right)$$

$$=\frac{1}{2}\left(\frac{x^2}{\sqrt{1+x^2}}+\sqrt{1+x^2}+\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}(\sqrt{1+x^2}+x)}\right)$$

$$=\frac{1}{2}\left(\frac{x^2}{\sqrt{1+x^2}}+\sqrt{1+x^2}+\frac{1}{\sqrt{1+x^2}}\right)\checkmark$$

13. **Graph** $r = \sin 2\theta$ **.**
See a different page. The picture has four leaves. The tips are on $(1, \pi/4)$ , $(-1, 3\pi/4)$ , $(1, 5\pi/4)$ , and $(-1, 7\pi/4)$ .

14. **Where does** $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{4^n n^2}$ **converge? Justify your answer.**
Use the ratio test. Let

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\frac{(x+2)^{n+1}}{4^{n+1}(n+1)^2}}{\frac{(x+2)^n}{4^n n^2}}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{4^{n+1}(n+1)^2}\,\frac{4^n n^2}{(x+2)^n}\right|$$

$$=\lim_{n\to\infty}\frac{|x+2|}{4}\left(\frac{n}{n+1}\right)^2 = \frac{|x+2|}{4}.$$

If $\frac{|x+2|}{4} < 1$, then $f(x)$ converges. If $1 < \frac{|x+2|}{4}$, then $f(x)$ diverges. Of course, $\frac{|x+2|}{4} < 1$ is the same as $|x+2| < 4$, which is $-4 < x+2 < 4$, which is $-6 < x < 2$. In a similar manner, we see that $1 < \frac{|x+2|}{4}$ occurs when $x < -6$ or $2 < x$. We study the end points.

We see that $f(2) = \sum_{n=1}^{\infty} \frac{4^n}{4^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. This is the $p$-series with $p = 2$. Since $1 < p$, we know that $f(2)$ converges.

We see that $f(-6) = \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. We apply the Absolute Convergence Test. The corresponding series of positive terms, which is $f(2)$, converges. Therefore $f(-6)$ converges. Thus,

$$\boxed{f(x) \text{ converges for } -6 \le x \le 2, \text{ and diverges elsewhere.}}$$

15. **Solve the initial value problem $\frac{dy}{dx} - \frac{y}{x} = 3x^2$ and $y(1) = 3$.**

Notice that the problem is a first order linear equation since it has the form $y' + P(x)y = Q(x)$. We multiple both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

The problem becomes

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 3x.$$

Notice that the left side is the derivative of $\frac{y}{x}$ with respect to $x$. Integrate both sides with respect to $x$ to get $\frac{y}{x} = \frac{3x^2}{2} + C$. Plug in $x = 1$ and $y = 3$ to see that $3 = \frac{3}{2} + C$; hence, $C = \frac{3}{2}$. We conclude that $\frac{y}{x} = \frac{3x^2}{2} + \frac{3}{2}$; that is, $\frac{y}{x} = \frac{3x^2+3}{2}$, or

$$\boxed{y = \frac{3x^3 + 3x}{2}}$$

Check. We see that $y(1) = \frac{3+3}{2} = 3$. Also,

$$\frac{dy}{dx} - \frac{y}{x} = \frac{9x^2+3}{2} - \frac{3x^2+3}{2} = 3x^2 \checkmark.$$

16. **Newton's law of cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium. Thus, if an object is taken from an oven at $300°$ F and left to cool in a room at $80°$ F, then its temperature $T$ after $t$ hours will satisfy the differential equation**

$$\frac{dT}{dt} = k(T - 80).$$

**If the temperature fell to $200°$ F after one hour, what will it be after 3 hours? (You may leave "ln" in your answer.)**

We know $T(0) = 300$ and $T(1) = 200$. We want $T(3)$. Separate the variables to get $\int \frac{dT}{T-80} = \int k\,dt$. Integrate to get $\ln|T-80| = kt + C$. Exponentiate to get $|T-80| = e^C e^{kt}$, or $T - 80 = \pm e^C e^{kt}$. Let $K = \pm e^C$. We now have $T - 80 = Ke^{kt}$. Plug in $t = 0$ to see that $300 - 80 = K$. Thus, $T - 80 = 220e^{kt}$. Plug in $t = 1$ to see that $200 - 80 = 220e^k$. So, $\ln \frac{6}{11} = k$. We now have $T(t) = 80 + 220e^{t \ln \frac{6}{11}}$. We conclude that

$$\boxed{T(3) = 80 + 220e^{3\ln\frac{6}{11}}\,{}^\circ\mathrm{F}}.$$