Math 142, Exam 4, Fall 2002

Name ________________________________

There are 11 problems on 6 pages. Problem 1 is worth 10 points. Each of the other problems is worth 9 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!**

1. **Find the limit of the sequence whose n^{th} term is** $a_n = n\sin\left(\frac{3}{n}\right)$**.**

We see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin\left(\frac{3}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{3}{n}\right)}{\frac{1}{n}}.$$

L'hopitals's rule applies since the top and the bottom both go to zero; so,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{-3}{n^2}\cos\left(\frac{3}{n}\right)}{\frac{-1}{n^2}} = \lim_{n\to\infty} 3\cos\left(\frac{3}{n}\right) = 3.$$

$$\boxed{\text{The sequence } \{a_n\} \text{ converges to 3.}}$$

2. **Does the series** $\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^n$ **converge? Justify your answer.**

The limit of the individual terms $\left(1-\frac{2}{n}\right)^n$ is equal to

$$\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n = e^{-2} \neq 0.$$

The Individual Term Test for Divergence tells me that

$$\boxed{\text{the series } \sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^n \text{ diverges.}}$$

3. **Consider the following sequence of numbers:** $a_2 = (1-\frac{1}{4})$, $a_3 = (1-\frac{1}{4})(1-\frac{1}{9})$, $a_4 = (1-\frac{1}{4})(1-\frac{1}{9})(1-\frac{1}{16})$, $\ldots$, $a_n = (1-\frac{1}{4})(1-\frac{1}{9})(1-\frac{1}{16})\ldots(1-\frac{1}{n^2})$, $\ldots$. **Does this infinite sequence converge? Justify your answer.**

This is a sequence of positive numbers. Observe that $a_{n+1} = a_n(1-\frac{1}{(n+1)^2})$. The number $1-\frac{1}{(n+1)^2}$ is less than 1; so $a_{n+1} < a_n$. We are studying a decreasing bounded sequence of real numbers . The completeness axiom assures us that

$$\boxed{\text{the sequence } \{a_n\} \text{ converges.}}$$

4. **Does the series** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ **converge? Justify your answer.**

We use the Alternating Series Test. Notice that the series alternates; $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$ for all n; and $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. The Alternating Series Test tells us that

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \text{ converges.}}$$

5. **Does the series** $\displaystyle\sum_{n=1}^{\infty} \frac{2\sqrt{n}}{n^2+1}$ **converge? Justify your answer.**

I do a limit comparison with the p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$. The p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ has $p = 3/2 > 1$; so $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. Observe that

$$\lim_{n\to\infty} \frac{\frac{2\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{2n^2}{n^2+1} = 2.$$

Two is a number. It is not zero or infinity; so the limit comparison test tells us that $\sum_{n=1}^{\infty} \frac{2\sqrt{n}}{n^2+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ both converge or both diverge. We have already seen that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. We conclude that

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \frac{2\sqrt{n}}{n^2+1} \text{ converges.}}$$

6. **Where does** $f(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}$ **converge? Justify your answer.**

Apply the ratio test. Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)3^{n+1}}}{\frac{(x-2)^n}{n3^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1} n3^n}{(x-2)^n(n+1)3^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{|x-2|}{3} \frac{n}{(n+1)} = \frac{|x-2|}{3}$$

Notice that $\rho < 1$ when $\frac{|x-2|}{3} < 1$; which is the same as $|x-2| < 3$; which is the same as $-3 < x-2 < 3$ or $-1 < x < 5$. If $-1 < x < 5$, then $f(x)$ converges. If $x < -1$ or $5 < x$, then $f(x)$ diverges. At -1,

$$f(-1) = \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Thus, $f(-1)$ is minus the sum of the Alternating Harmonic series, which we know converges. At 5,

$$f(5) = \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n};$$

which is the harmonic series which diverges. We conclude that

the function $f(x)$ converges for $-1 \leq x < 5$ and diverges everywhere else.

7. **Find** $\lim_{x\to 0} \frac{\sin(x^2) - x^2 + \frac{x^6}{6}}{x^{10}}$**.**

We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots.$$

It follows that

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots,$$

and

$$\sin(x^2) - x^2 + \frac{x^6}{6} = \frac{x^{10}}{5!} - \dots.$$

Thus,

$$\lim_{x\to 0} \frac{\sin(x^2) - x^2 + \frac{x^6}{6}}{x^{10}} = \lim_{x\to 0} \frac{\frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \dots}{x^{10}}$$

$$= \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^4}{7!} + \frac{x^8}{9!} - \dots\right) = \frac{1}{5!} = \boxed{\frac{1}{120}.}$$

8. **Which familair function is equal to**
$f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{3!} + \frac{x^6}{4!} + \frac{x^8}{5!} + \frac{x^{10}}{6!} + \dots$ **? Explain.**

Notice that the exponent is "almost" twice the denominator if one forgets about the factorial. In other words,

$$x^2 f(x) = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \frac{x^{12}}{6!} + \dots = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}.$$

Thus,

$$1 + x^2 f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

We know that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$. It follows that

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

In other words,

$$1 + x^2 f(x) = e^{x^2},$$

and

$$\boxed{f(x) = \frac{e^{x^2} - 1}{x^2}.}$$

9. **Approximate $e^{\frac{-1}{10}}$ with an error at most 10^{-3} . Explain what you are doing.**

We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. It follows that

$$e^{\frac{-1}{10}} = 1 - \frac{1}{10} + \frac{1}{2 \times 10^2} - \frac{1}{3! \times 10^3} + \dots .$$

This is an alternating series. We know that $\frac{1}{n! \times 10^n} > \frac{1}{(n+1)! \times 10^{n+1}}$. It is clear that $\lim_{n \to \infty} \frac{1}{n! \times 10^n} = 0$. The Alternating series test applies. The first term which is less than 10^{-3} is $\frac{1}{3! \times 10^3}$. We conclude that

$$\boxed{1 - \frac{1}{10} + \frac{1}{2 \times 10^2} \text{ approximates } e^{\frac{-1}{10}} \text{ with an error of at most } 10^{-3}.}$$

10. **Approximate** $\sum\limits_{n=10}^{\infty} \frac{1}{n^2}$ **. Your approximation should be "close to" but more than the actual value. Explain what you are doing.**

I will use an improper integral to approximate $\sum\limits_{n=10}^{\infty} \frac{1}{n^2}$. Draw the graph of $y = \frac{1}{x^2}$. Approximate the area under the curve starting at $x = 9$ by drawing rectangles with base from one integer to the next and height given by the y-coordinate on the curve above the right hand endpoint. I put a picture on a separate piece of paper. The area inside the boxes underestimates the area under the curve. The area inside the boxes is $\sum\limits_{n=10}^{\infty} \frac{1}{n^2}$. The area under the curve is $\int\limits_9^{\infty} x^{-2} dx$. Thus,

$$\sum_{n=10}^{\infty} \frac{1}{n^2} \leq \int_9^{\infty} x^{-2} dx = \lim_{b \to \infty} \left. \frac{-1}{x} \right|_9^b$$

$$= \lim_{b \to \infty} \frac{-1}{b} + \frac{1}{9} = \frac{1}{9}.$$

We conclude that

$$\boxed{\sum_{n=10}^{\infty} \frac{1}{n^2} \leq \frac{1}{9}.}$$

11. **A ball is dropped from a height of 100 feet. Each time it bounces, it rebounds to $\frac{4}{5}$ its previous height. Find the total distance it travels before coming to rest.**

The ball goes down 100 feet, up $\left(\frac{4}{5}\right) 100$ feet, down $\left(\frac{4}{5}\right) 100$ feet, up $\left(\frac{4}{5}\right)^2 100$ feet, down $\left(\frac{4}{5}\right)^2 100$ feet, etc. In total the ball travels

$$100 + \left(\tfrac{4}{5}\right) 100 + \left(\tfrac{4}{5}\right) 100 + \left(\tfrac{4}{5}\right)^2 100 + \left(\tfrac{4}{5}\right)^2 100 + \left(\tfrac{4}{5}\right)^3 100 + \left(\tfrac{4}{5}\right)^3 100 + \ldots$$

$$= 100 + \left[\left(\tfrac{4}{5}\right) 200 + \left(\tfrac{4}{5}\right)^2 200 + \left(\tfrac{4}{5}\right)^3 200 + \ldots \right]$$

feet. The expression inside the brackets is the geometric series with $r = \frac{4}{5}$ and $a = \left(\frac{4}{5}\right) 200$. This geometric series converges to $\frac{a}{1-r} = \frac{\left(\frac{4}{5}\right)200}{\frac{1}{5}} = 800$. The ball travels $100 + 800 = \boxed{900 \text{ feet}}$.