Exam 3 Fall 2002 Math 142

Name ____

There are 10 problems on 5 pages. Each problem is worth 10 points. each. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!** CHECK your answer whenever possible.

1. **Find the limit of the sequence whose** n^{th} **term is** $a_n = n\sin\left(\frac{1}{2n}\right)$**.**

We see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin\left(\frac{1}{2n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{2n}\right)}{\frac{1}{n}}.$$

The top and the bottom both go to zero, so l'Hopital's rule tells us that this limit is equal to

$$= \lim_{n\to\infty} \frac{\frac{-1}{2n^2}\cos\left(\frac{1}{2n}\right)}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{1}{2}\cos\left(\frac{1}{2n}\right) = \frac{1}{2}.$$

We conclude that

$$\boxed{\text{the sequence converges to } \frac{1}{2}.}$$

2. **Find the limit of the sequence whose** n^{th} **term is** $a_n = \left(\frac{n-1}{n+1}\right)^n$**.**

We see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{2}{n+1}\right)^n = \lim_{n\to\infty} \frac{\left(1-\frac{2}{n+1}\right)^{n+1}}{1-\frac{2}{n+1}}.$$

Let $m = n+1$. Our limit is equal to

$$\lim_{m\to\infty} \frac{\left(1-\frac{2}{m}\right)^m}{1-\frac{2}{m}}.$$

We know that

$$\lim_{x\to\infty} \left(1+\frac{r}{x}\right)^x = e^r.$$

We conclude that our limit is equal to $\frac{e^{-2}}{1}$. Thus,

$$\boxed{\text{the sequence converges to } e^{-2}.}$$

3. **Find** $\int_0^\infty \frac{1}{1+x^2}\,dx$**.**

The problem is equal to

$$\lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \arctan x \Big|_0^b = \lim_{b\to\infty} \arctan(b) - \arctan(0) = \pi/2 - 0 = \boxed{\frac{\pi}{2}}.$$

4. **Find** $\int_{-3}^{1} \frac{1}{x^2}\,dx$ **.**

The function $\frac{1}{x^2}$ goes to infinity at $x = 0$. It is absolutely necessary to realize that this is an improper integral. A picture is included on another page. The integral is equal to

$$\lim_{b\to 0^-} \int_{-3}^{b} \frac{1}{x^2}\,dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x^2}\,dx = \lim_{b\to 0^-} \left.\frac{-1}{x}\right|_{-3}^{b} + \lim_{a\to 0^+} \left.\frac{-1}{x}\right|_{a}^{1}$$

$$= \lim_{b\to 0^-} \frac{-1}{b} - \frac{-1}{-3} + \lim_{a\to 0^+} \frac{-1}{1} - \frac{-1}{a} = +\infty - \frac{-1}{-3} - 1 + \infty.$$

$$\boxed{\text{the integral diverges to } +\infty.}$$

By the way, the picture confirms that the integral represents an area. The answer can not possibly be zero or any negative number.

5. **Find** $\int \frac{1}{\sqrt{4-9x^2}}\,dx$ **. Check your answer.**

This integral is equal to

$$\int \frac{1}{2\sqrt{1-\frac{9x^2}{4}}}\,dx.$$

Let $u = 3x/2$. We have $du = (3/2)dx$ and the integral is equal to

$$\frac{2}{2\cdot 3} \int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\arcsin u + C = \boxed{\frac{1}{3}\arcsin(3x/2) + C.}$$

CHECK: The derivative of our proposed answer is

$$\frac{1}{3}\frac{\frac{3}{2}}{\sqrt{1-\frac{9x^2}{4}}} = \frac{1}{2\sqrt{1-\frac{9x^2}{4}}} = \frac{1}{\sqrt{4-9x^2}}.\checkmark$$

6. **Find** $\int \frac{\ln x}{x^2}\,dx$ **. Check your answer.**

We use integration by parts. Let $u = \ln x$ and $dv = x^{-2}$. We calculate that $du = \frac{1}{x}$ and $v = \frac{-1}{x}$. The original integral is equal to

$$\frac{-\ln x}{x} + \int x^{-2}\,dx = \boxed{\frac{-\ln x}{x} + \frac{-1}{x} + C}.$$

CHECK: The derivative of our proposed answer is

$$-\ln x(-\frac{1}{x^2}) - \frac{1}{x^2} + \frac{1}{x^2} = \frac{\ln x}{x^2}.\checkmark$$

7. **Find** $\displaystyle\int \frac{4x^2+x-2}{x^2(x-1)}\,dx$ **. Check your answer.**

Let

$$\frac{4x^2+x-2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Multiply both sides by $x^2(x-1)$.

$$4x^2+x-2 = Ax(x-1) + B(x-1) + Cx^2$$

$$4x^2+x-2 = (A+C)x^2 + (B-A)x - B.$$

We see that $B=2$; $1=B-A$ (so $A=1$); and $4=A+C$ (so $C=3$). The original problem is equal to

$$\int \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x-1}\,dx = \boxed{\ln|x| - \frac{2}{x} + 3\ln|x-1| + C.}$$

CHECK: The derivative of our proposed answer is

$$\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x-1} = \frac{x(x-1)+2(x-1)+3x^2}{x^2(x-1)} = \frac{4x^2+x-2}{x^2(x-1)}.\checkmark$$

8. **Find** $\displaystyle\int \frac{3x^2-3x+1}{x(x^2+1)}\,dx$ **. Check your answer.**

Let

$$\frac{3x^2-3x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiply both sides by $x(x^2+1)$ to get

$$3x^2-3x+1 = A(x^2+1) + (Bx+C)x$$

$$3x^2-3x+1 = (A+B)x^2 + Cx + A.$$

We see that $A=1$, $C=-3$, and $A+B=3$ (so, $B=2$). The original problem is equal to

$$\int \frac{1}{x} + \frac{2x-3}{x^2+1}\,dx = \boxed{\ln|x| + \ln(x^2+1) - 3\arctan x + C}.$$

CHECK: The derivative of our proposed answer is

$$\frac{1}{x} + \frac{2x}{x^2+1} - \frac{3}{x^2+1} = \frac{x^2+1+2x^2-3x}{x^2+1} = \frac{3x^2-3x+1}{x^2+1}.\checkmark$$

9. **Find the general solution of** $\displaystyle\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$ **. Check your answer.**

This problem is in the form $y' + P(x)y = Q(x)$ Let

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{\frac{3}{2}}.$$

Multiply both sides of the original problem by $\mu(x)$ to get

$$(x^2+1)^{\frac{3}{2}}\frac{dy}{dx} + (x^2+1)^{\frac{1}{2}}3xy = 6x(x^2+1)^{\frac{1}{2}}.$$

Notice that the left side is now equal to

$$\frac{d}{dx}\left((x^2+1)^{\frac{3}{2}}y\right).$$

Integrate both sides with respect to x to get

$$(x^2+1)^{\frac{3}{2}}y = \int 6x(x^2+1)^{\frac{1}{2}}\,dx = 2(x^2+1)^{\frac{3}{2}} + C.$$

So,

$$\boxed{y = 2 + C(x^2+1)^{-\frac{3}{2}}.}$$

CHECK. We compute that

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = -C\frac{3}{2}2x(x^2+1)^{-\frac{5}{2}} + \frac{3x}{x^2+1}(2 + C(x^2+1)^{-\frac{3}{2}})$$

$$= \frac{-3Cx}{(x^2+1)^{\frac{5}{2}}} + \frac{6x}{x^2+1} + \frac{3Cx}{(x^2+1)^{\frac{5}{2}}} = \frac{6x}{x^2+1}.\checkmark$$

10. **Which number** $\int_1^{n+1}\frac{1}{\sqrt{x}}\,dx$ **or** $\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$ **is bigger? Does the sequence whose** n^{th} **term is** $a_n = \sum_{k=1}^{n}\frac{1}{\sqrt{k}}$ **converge? Justify your answer.**

I have drawn a picture on a separate page. The integral is equal to the area under the curve $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = n+1$. The sum is equal to the area inside n boxes which OVER estimate the area under the curve. We conclude that the sum is $\boxed{\text{LARGER}}$ than the integral. The limit as n goes to infinity of the integral is

$$\lim_{n\to\infty} 2\sqrt{x}\Big|_1^{n+1} = \lim_{n\to\infty} 2\sqrt{n+1} - 2\sqrt{1} = \infty.$$

We see that

$$\lim_{n\to\infty} a_n > \lim_{n\to\infty}\int_1^{n+1}\frac{1}{\sqrt{x}}\,dx = +\infty.$$

We conclude that

$$\boxed{\text{the sequence } \{a_n\} \text{ diverges to } +\infty.}$$