Fall 2001, Exam 4, Math 142, The solution to problems 3, 4, 9, 10.

3. Does $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ converge? Justify your answer.

The n^{th} partial sum of this series is

$$s_m = \sum_{n=1}^{m} \ln\left(\frac{n}{n+1}\right) = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \dots + \ln\left(\frac{m}{m+1}\right) =$$
$$\ln\left(\frac{1}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{m}{m+1}\right) = -\ln(m+1).$$

The limit as m goes to ∞ of the m^{th} partial sum is $\lim_{m\to\infty} -\ln(m+1) = -\infty$.

The series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ DIVERGES

to $-\infty$.

4. Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converge? Justify your answer.

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the p-series with $p = 2 > 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ CONVERGES

by the absolute convergence test.

9. Which familiar function is equal to $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$? Justify your answer.

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \textbf{for } -1 < x < 1\,.$$

Replace x by $-x$ to see that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad \textbf{for } -1 < x < 1\,.$$

Integrate

$$\ln|1+x| = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \textbf{for } -1 < x < 1$$

for some constant C. Plug in $x=0$ to see that $0=\ln 1 = C$. Notice that if $-1<x<1$, then $0<x+1$; so, $|x+1| = x+1$. We conclude that

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \quad \text{for } -1<x<1.}$$

10. Approximate $\int_0^{1/10} e^{-x^2}\,dx$ with an error of at most 10^{-8}. Justify your answer.

We know that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots .$$

Therefore,

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

and

$$\int_0^{1/10} e^{-x^2}\,dx = \int_0^{1/10} 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots\, dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{2\cdot 5} - \frac{x^7}{3!\cdot 7} + \frac{x^9}{4!\cdot 9} + \dots \Big|_0^{1/10}$$

$$= \frac{1}{10} - \frac{1}{3\cdot 10^3} + \frac{1}{2\cdot 5\cdot 10^5} - \frac{1}{3!\cdot 7\cdot 10^7} + \frac{1}{4!\cdot 9\cdot 10^9} + \dots$$

This is an alternating series. The absolute value of the terms decreases and has limit zero. The alternating series test applies. The series converges and the first term which is less than $\frac{1}{10^8}$ **is** $\frac{1}{3!\cdot 7\cdot 10^7}$. **We conclude that**

$$\boxed{\frac{1}{10} - \frac{1}{3\cdot 10^3} + \frac{1}{2\cdot 5\cdot 10^5} \quad \textbf{approximates} \int_0^{1/10} e^{-x^2}\,dx. \quad \textbf{with an error less than } 10^{-8}}$$