

Exam 1, Fall 2001, Math 142 Solutions to problems 4 and 5.

4. Find the volume of the solid generated by revolving the region bounded by $y = e^x$, the x -axis, the y -axis, and $x = 1$ about the x -axis.

Spin the rectangle. Get a disc of volume $\pi r^2 t$, where $t = dx$, and r is equal to the y -coordinate, which is e^x . So the volume is

$$\pi \int_0^1 e^{2x} dx = \pi \frac{e^{2x}}{2} \Big|_0^1 = \boxed{\frac{\pi}{2}(e^2 - 1)}.$$

The picture is on a separate page.

5. Let $f(x) = x^2 + 2x$ for $x \leq -1$. Find $f^{-1}(x)$. What is the domain of $f^{-1}(x)$? Verify that $f(f^{-1}(x)) = x$ for all x in the domain of $f^{-1}(x)$. Verify that $f^{-1}(f(x)) = x$ for all x in the domain of $f(x)$. Graph $y = f(x)$. Graph $y = f^{-1}(x)$.

Let $y = f^{-1}(x)$. So $f(y) = x$ and $y \leq -1$. It follows that $y^2 + 2y = x$. I could use the quadratic formula. I will just complete the square.

$$\begin{aligned}(y + 1)^2 &= x + 1 \\ y + 1 &= \pm \sqrt{x + 1} \\ y &= -1 \pm \sqrt{x + 1}.\end{aligned}$$

But $y \leq -1$; so, $y = -1 - \sqrt{x + 1}$. We conclude that

$$\boxed{f^{-1}(x) = -1 - \sqrt{x + 1} \quad \text{for } -1 \leq x.}$$

The picture appears on a separate page. We check our answer. Take $-1 \leq x$. We see that

$$\begin{aligned}f(f^{-1}(x)) &= f(-1 - \sqrt{x + 1}) = (-1 - \sqrt{x + 1})^2 + 2(-1 - \sqrt{x + 1}) \\ &= 1 + 2\sqrt{x + 1} + (\sqrt{x + 1})^2 - 2 - 2\sqrt{x + 1} = (x + 1) - 1 = x.\checkmark\end{aligned}$$

Take $x \leq -1$.

$$f^{-1}(f(x)) = f^{-1}(x^2 + 2x) = -1 - \sqrt{x^2 + 2x + 1} = -1 - \sqrt{(x + 1)^2} = -1 - |x + 1|.$$

We know that $x \leq -1$; so, $x + 1 \leq 0$. Thus $|x + 1| = -(x + 1)$, and

$$-1 - |x + 1| = -1 - (-(x + 1)) = -1 + x + 1 = x.\checkmark$$

The picture appears on a separate page.