Final Exam, Math 142, Fall 2000, problems 7, 8, 9, 10, 11, and 12

7. Find $\displaystyle\int \frac{-x^2+5x-2}{(x-1)(x^2+1)}\,dx$.

Set

$$\frac{-x^2+5x-2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Multiply both sides by $(x-1)(x^2+1)$ **to get**

$$-x^2+5x-2 = A(x^2+1) + (Bx+C)(x-1);$$

hence,

$$-x^2+5x-2 = (A+B)x^2 + (-B+C)x + (A-C).$$

Equate the corresponding coefficients:

$$\begin{aligned} -1 &= A+B \\ 5 &= -B+C, \text{ and} \\ -2 &= A-C. \end{aligned}$$

Replace Equation 3 by Equation 3 minus Equation 1:

$$\begin{aligned} -1 &= A+B \\ 5 &= -B+C, \text{ and} \\ -1 &= -B-C. \end{aligned}$$

Replace Equation 3 by Equation 3 plus Equation 2:

$$\begin{aligned} -1 &= A+B \\ 5 &= -B+C, \text{ and} \\ 4 &= -2B. \end{aligned}$$

So, $B=-2$ **,** $C=3$ **, and** $A=1$ **. Thus,**

$$\int \frac{-x^2+5x-2}{(x-1)(x^2+1)}\,dx = \int \frac{1}{x-1} + \frac{-2x+3}{x^2+1}\,dx$$
$$= \boxed{\ln|x-1| - \ln(x^2+1) + 3\arctan x + C}.$$

Of course, the derivative of the proposed answer is

$$\frac{1}{x-1} - \frac{2x}{x^2+1} + \frac{3}{1+x^2} = \frac{(x^2+1) - 2x(x-1) + 3(x-1)}{(x-1)(x^2+1)} = \frac{-x^2+5x-2}{(x-1)(x^2+1)}.\checkmark$$

8. Find $\displaystyle\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$.

We have

$$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{(x^2+2)^2}\,dx = \lim_{b\to\infty} \frac{-1}{2(x^2+2)}\bigg|_0^b$$

$$= \lim_{b\to\infty}\frac{-1}{2(b^2+2)} + \frac{1}{4} = \boxed{\frac{1}{4}}.$$

9. Find $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$.
The top and the bottom both go to 0. We use L'Hopital's rule to see that

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x}.$$

L'Hopital's rule applies once again. We get

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \lim_{x\to 0}\frac{\cos x}{2} = \boxed{\frac{1}{2}}.$$

10. Find $\lim_{x\to \pi}\dfrac{1-\cos x}{x^2}$.
It is clear that

$$\lim_{x\to \pi}\frac{1-\cos x}{x^2} = \boxed{\frac{2}{\pi^2}.}$$

11. Let $f(x) = x^3 e^{\sin(x^2)}$. Find $f'(x)$.
We see that

$$\boxed{f'(x) = x^3(2x)\cos(x^2)e^{\sin(x^2)} + 3x^2 e^{\sin(x^2)}.}$$

12. Let $f(x) = \frac{17}{\arcsin x}$. Find $f'(x)$.
We see that

$$\boxed{f'(x) = \frac{-17}{(\arcsin x)^2}\frac{1}{\sqrt{1-x^2}}.}$$