Math 142, Fall 2000, Exam 4, Problems 9, 10, and 11

9. Consider the series $\displaystyle\sum_{k=4}^{\infty} \frac{1}{k} - \frac{1}{k+1}$. Give a closed formula for the partial sum $s_n = \displaystyle\sum_{k=4}^{n} \frac{1}{k} - \frac{1}{k+1}$. Does the series converge? If so, what is the sum of the series?

We see that

$$s_n = \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= \left(\frac{1}{4} - \frac{1}{n+1}\right).$$

We also see that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{n+1}\right) = \frac{1}{4}$. **Thus,**

$$\boxed{\textbf{The series converges to } \frac{1}{4}.}$$

10. Find $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2}$.

We have

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^{b} \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} x \Big|_0^b = \lim_{b\to\infty} \tan^{-1} b - \tan^{-1}(0)$$
$$= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}.}$$

11. Find $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$.

Use L'hopitals' rule four times to get that the limit is $\frac{1}{24}$; OR use

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\ .$$

Thus,

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \lim_{x\to 0} \frac{(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots) - 1 + \frac{x^2}{2}}{x^4}$$
$$= \lim_{x\to 0} \frac{+\frac{x^4}{4!} - \frac{x^6}{6!} + \dots}{x^4} = \lim_{x\to 0} +\frac{1}{4!} - \frac{x^2}{6!} + \cdots = \boxed{\frac{1}{4!}}.$$