Fall 2000 Math 142 Exam 4

85

PRINT Your Name:_____________________________

There are 11 problems on 6 pages. Problem 1 is worth 10 points. Each of the other problems is worth 9 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **NO CALCULATORS!**

1. Find the Taylor polynomial of degree six, $P_6(x)$, for $f(x) = \sin x$ about $a = 0$

$$P_6(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6$$

$f(x) = \sin x \quad f(0) = 0$

$f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0$

$f'''(x) = -\cos x \quad f'''(0) = -1$

$f^{(4)}x = \sin x \quad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$

$f^{(6)}(x) = -\sin x \quad f^{(6)}(0) = -1$

$f^{(7)}(x) = -\cos x$

$$P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

2. Use your answer to problem [redacted] to estimate $\int_0^1 \frac{\sin x}{x}\,dx$. How good is your estimate? Explain.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + R_6(x)$$

$$R_6(x) = \frac{f^{(7)}(c)}{7!}x^7$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{f^{(7)}(c)}{7!}x^6$$

$$\left|\frac{\sin x}{x} - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right)\right| = \left|\frac{f^{(7)}(c)}{7!}x^6\right| \le \frac{|x|^6}{7!}$$

$$\int_0^1 \frac{\sin x}{x}\,dx \approx \int_0^1 1 - \frac{x^2}{3!} + \frac{x^4}{5!}\,dx = x - \frac{x^3}{3!\cdot 3} + \frac{x^5}{5!5}\Big]_0^1$$

$$= 1 - \frac{1}{3!3} + \frac{1}{5!5}$$

$$\left|\int_0^1 \frac{\sin x}{x}\,dx - \int_0^1 1 - \frac{x^2}{3!} + \frac{x^4}{5!}\,dx\right| \le \int_0^1 \left|\frac{\sin x}{x} - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right)\right|dx$$

$$< \int_0^1 \frac{x^6}{7!}\,dx = \frac{x^7}{7!7}\Big]_0^1 = \frac{1}{7!7}$$

So $\int_0^1 \frac{\sin x}{x}\,dx$ is approximately equal to [redacted] $- \frac{1}{3!3} + \frac{1}{5!5}$ and the error that this approximation introduces is at most $\frac{1}{7!7}$