Math 142, Fall 2000, Exam 2, Solution to Problems 3, 4, 7, and 8

3. Find $\displaystyle\int \sin 4x\ \sin 5x\, dx$.

Recall that

$$\begin{aligned}\cos(A-B) &= \cos A\cos B + \sin A\sin B\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B\end{aligned}$$

The top equation minus the bottom equation gives us

$$\frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) = \sin A\sin B.$$

Take $A = 4x$ **and** $B = 5x$ **. (Keep in mind that** $\cos(-x) = \cos x$ **.) The original integral is equal to**

$$\frac{1}{2}\int \cos x - \cos(9x)\, dx = \boxed{\frac{1}{2}\left(\sin x - \frac{\sin 9x}{9}\right) + C}.$$

4. Find $\displaystyle\int \csc^3 x\, dx$.

Use integration by parts with

$$\begin{aligned}u &= \csc x & v &= -\cot x\\ du &= -\csc x\cot x\, dx & dv &= \csc^2 x\, dx\end{aligned}$$

to see that

$$\int \csc^3 x\, dx = -\csc x\cot x - \int \csc x\cot^2 x\, dx.$$

Of course,

$$\sin^2 x + \cos^2 x = 1.$$

Divide both sides by $\sin^2 x$ **to get**

$$1 + \cot^2 x = \csc^2 x.$$

It follows that

$$\begin{aligned}&\int \csc^3 x\, dx = -\csc x\cot x - \int \csc x(\csc^2 x - 1)\, dx\\ &= -\csc x\cot x - \int \csc^3 x\, dx + \int \csc x\, dx.\end{aligned}$$

Add $\int \csc^3 x\, dx$ **to both sides to see that**

$$\begin{aligned}&2\int \csc^3 x\, dx = -\csc x\cot x + \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x}\, dx\\ &= -\csc x\cot x - \ln|\csc x + \cot x|.\end{aligned}$$

Conclude that

$$\boxed{\int \csc^3 x\,dx = \frac{1}{2}\left(-\csc x\cot x - \ln|\csc x + \cot x|\right) + C.}$$

We check our answer:

$$\frac{d}{dx}\left(\frac{1}{2}\left(-\csc x\cot x - \ln|\csc x + \cot x|\right)\right)$$
$$= \frac{1}{2}\left(-\csc x(-\csc^2 x) + \cot x(\csc x\cot x) - \frac{-\csc x\cot x - \csc^2 x}{\csc x + \cot x}\right)$$
$$= \frac{1}{2}\left(\csc^3 x + \csc x(\csc^2 x - 1) - \frac{-\csc x(\cot x + \csc x)}{\csc x + \cot x}\right)$$
$$= \frac{1}{2}\left(\csc^3 x + \csc^3 x - \csc x + \csc x\right) \checkmark$$

7. Find $\int \frac{1}{x^2+4x+5}\,dx$.
We see that

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{1+(x+2)^2}\,dx = \boxed{\arctan(x+2) + C}.$$

Of course, the derivative of $\arctan(x+2)$ **is equal to** $\frac{1}{1+(x+2)^2}$ **.**

8. Find $\int \frac{\sqrt{x^2-4}}{x}\,dx$.
We will make a trig substitution. Let $x = 2\sec\theta$ **. It follows that**

$$dx = 2\sec\theta\tan\theta d\theta$$
$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\tan\theta.$$

We have

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{4\tan\theta\sec\theta\tan\theta}{2\sec\theta}d\theta$$
$$= \int 2\tan^2\theta d\theta = 2\int(\sec^2\theta - 1)d\theta = 2(\tan\theta - \theta) + C$$
$$= \boxed{\sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right) + C.}$$

The derivative of our proposed answer is

$$\frac{x}{\sqrt{x^2-4}} - 2\frac{\frac{1}{2}}{\frac{x}{2}\sqrt{(\frac{x}{2})^2-1}} = \frac{x}{\sqrt{x^2-4}} - \frac{2}{x\sqrt{(\frac{x}{2})^2-1}} = \frac{x}{\sqrt{x^2-4}} - \frac{2}{x\frac{1}{2}\sqrt{x^2-4}}$$
$$= \frac{x^2}{x\sqrt{x^2-4}} - \frac{4}{x\sqrt{x^2-4}} = \frac{x^2-4}{x\sqrt{x^2-4}}.\checkmark$$