Math 142 Exam 3 Solutions Fall 2004

PRINT Your Name:______________________________

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office after I have graded it. (I will send you an e-mail when I am finished with it.) You may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 6:00 PM today.

1. **Find** $\int \cos^3 x\,dx$ **. Check your answer.**

Save $\cos x$. Convert the other two $\cos x$'s to $\sin x$ using $\cos^2 x = 1 - \sin^2 x$. Let $u = \sin x$. It follows that $du = \cos x\,dx$. We see that

$$\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1 - \sin^2 x)\cos x\,dx = \int (1 - u^2)\,du$$

$$= u - \frac{u^3}{3} + C = \boxed{\sin x - \frac{\sin^3 x}{3} + C}.$$

Check: The derivative of the proposed answer is

$$\cos x - \sin^2 x(\cos x) = \cos x(1 - \sin^2 x) = \cos^3 x. \checkmark$$

2. **Find** $\int \cos^2 x\,dx$ **.**

We see that

$$\int \cos^2 x\,dx = \frac{1}{2}\int (1 + \cos 2x)\,dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$$

3. **Find** $\int x \ln x\,dx$ **. Check your answer.**

We apply integration by parts: $\int u\,dv = uv - \int v\,du$ with $u = \ln x$ and $dv = x\,dx$. We see that $du = \frac{1}{x}dx$, $v = \frac{x^2}{2}$, and

$$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C.}$$

Check: The derivative of the proposed answer is

$$\frac{x^2}{2}\frac{1}{x} + x \ln x - \frac{x}{2} = x \ln x. \checkmark$$

4. **Find** $\displaystyle\int \frac{x+1}{(x-3)^2}\,dx$ **. Check your answer.**

Multiply both sides of

$$\frac{x+1}{(x-3)^2} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2}$$

by $(x-3)^2$ to see that

$$x+1 = A(x-3) + B.$$

It follows that

$$x+1 = Ax + (-3A+B).$$

Equate the corresponding coefficients to see that $A = 1$ and $-3A+B = 1$. Thus, $B = 4$. We verify that

$$\frac{1}{(x-3)} + \frac{4}{(x-3)^2} = \frac{(x-3)+4}{(x-3)^2} = \frac{x+1}{(x-3)^2}. \checkmark$$

Now we do the integral:

$$\int \frac{x+1}{(x-3)^2}\,dx = \int \frac{1}{(x-3)} + \frac{4}{(x-3)^2}\,dx = \boxed{\ln|x-3| - \frac{4}{x-3} + C}.$$

5. **Find** $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$ **. Check your answer.**

We will use trig substitution. Let $x = \tan\theta$. It follows that $dx = \sec^2\theta\,d\theta$ and $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$. Thus,

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln|\sqrt{1+x^2} + x| + C}.$$

Check: The derivative of $\ln(\sqrt{1+x^2}+x)$ is

$$\frac{\frac{2x}{2\sqrt{1+x^2}} + 1}{\sqrt{1+x^2}+x} = \frac{\frac{x}{\sqrt{1+x^2}} + 1}{\sqrt{1+x^2}+x} = \frac{x+\sqrt{1+x^2}}{(\sqrt{1+x^2}+x)\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}. \checkmark$$

6. **Find** $\displaystyle\lim_{x\to\frac{\pi}{4}} \frac{\sin x}{x}$ **.**

The answer is $\boxed{\frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}}.}$

7. **Find the limit of the sequence whose n^{th} term is $a_n = \left(\frac{n+3}{n}\right)^n$.**

The answer is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+3}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = \boxed{e^3}$$

because we know that $\lim\limits_{n\to\infty} (1+\frac{r}{n})^n = e^r$.

8. **Find** $\int_1^3 \frac{1}{(x-2)^2}\,dx$ **.**

Look at the graph. This integral represents the area between a curve and the x-axis. The curve is ABOVE the x-axis. The integral is either a positive number or $+\infty$. If your answer is a negative number, then your answer is total nonsense and you will recieve ZERO points.

$$\int_1^3 \frac{1}{(x-2)^2}\,dx = \lim_{b\to 2^-} \int_1^b \frac{1}{(x-2)^2}\,dx + \lim_{a\to 2^+} \int_a^3 \frac{1}{(x-2)^2}\,dx$$

$$= \lim_{b\to 2^-} -(x-2)^{-1}\Big|_1^b + \lim_{a\to 2^+} -(x-2)^{-1}\Big|_a^3$$

$$= \lim_{b\to 2^-} \frac{-1}{(b-2)} + \frac{1}{(1-2)} + \lim_{a\to 2^+} \frac{-1}{(3-2)} + \frac{1}{(a-2)} = +\infty - 1 + \infty - 1 = \boxed{+\infty}.$$

9. **Does the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge or diverge? If the series converges, what is its sum? Explain your answer.**

This series $\boxed{\text{diverges}}$. Indeed, the picture shows that the partial sum $s_n = \sum_{k=1}^{n} \frac{1}{k}$ is at least as large as

$$\int_1^{n+1} \frac{1}{x}\,dx = \ln x\Big|_1^{n+1} = \ln(n+1).$$

The sum of the series is the limit of the sequence of partial sums and it is clear that $\lim\limits_{n\to\infty} s_n \geq \lim\limits_{n\to\infty} \ln(n+1) = \infty$.

10. **Does the series $\sum_{k=1}^{\infty} \frac{4}{7^k}$ converge or diverge? If the series converges, what is its sum? Explain your answer.**

This series is the geometric series with initial term $a = \frac{4}{7}$ and ratio $r = \frac{1}{7}$. We see that $-1 < r < 1$. It follows that the geometric series converge to the sum $\frac{a}{1-r} = \boxed{\frac{\frac{4}{7}}{1-\frac{1}{7}}} = \frac{4}{7-1} = \frac{2}{3}$.