Math 142, Exam 2, Fall 2009 Solutions

PRINT Your Name:______________________

When does your FRIDAY class meet? ______
Write your answers as legibly as you can.

There are 10 problems on 5 pages. The exam is worth 100 points. Each problem is worth 10 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website a few hours after the exam is finished.

1. **Find** $\displaystyle\int \frac{e^x\,dx}{1+e^x}$. **Check your answer.**

Let $u = e^x$. It follows that $du = e^x\,dx$. The integral is

$$\int \frac{du}{u} = \ln|u| + C = \boxed{\ln(1+e^x)+C.}$$

Check: The derivative of the proposed answer is $\frac{e^x}{1+e^x}$. ✓

2. **Compute** $\displaystyle\int_{-1}^{3} \frac{1}{x^2}dx$

The function $f(x) = \frac{1}{x^2}$ goes to infinity as x approaches zero. For all x other than zero, $f(x)$ is positive. Thus, this integral represents an area. Either this integral is infinite; or else, the integral is finite and positive.

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \lim_{b\to 0^-}\int_{-1}^{b} \frac{1}{x^2}\,dx + \lim_{a\to 0^+}\int_{a}^{3} \frac{1}{x^2}\,dx$$

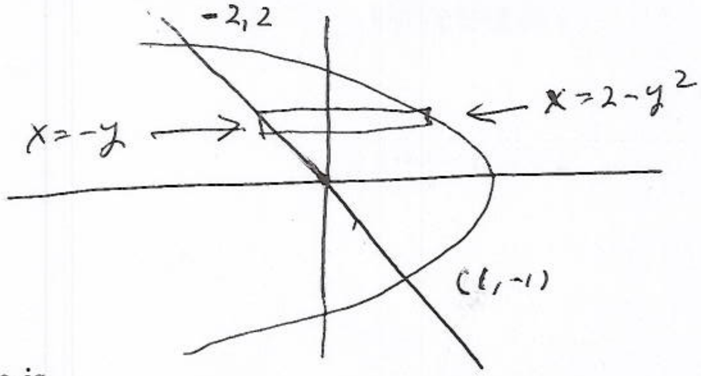
$$= \lim_{b\to 0^-} \left.\frac{-1}{x}\right|_{-1}^{b} + \lim_{a\to 0^+} \left.\frac{-1}{x}\right|_{a}^{3} = \lim_{b\to 0^-}\left(\frac{-1}{b} - \frac{-1}{-1}\right) + \lim_{a\to 0^+}\left(\frac{-1}{3} - \frac{-1}{a}\right).$$

Notice that

$$\lim_{b\to 0^-}\left(\frac{-1}{b}\right) = +\infty \quad \text{and} \quad \lim_{a\to 0^+}\left(-\frac{-1}{a}\right) = +\infty.$$

Thus, $\boxed{\text{the integral diverges to } +\infty.}$ Notice that $-\frac{4}{3}$ has NOTHING TO DO WITH THE FINAL ANSWER. An "answer" $-\frac{4}{3}$ will recieve a score of 0.

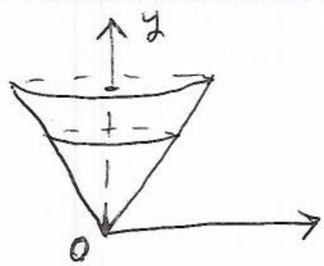
3. **Find the area between $x + y = 0$ and $2 = x + y^2$.**



The area is

$$\int_{-1}^{2} 2 - y^2 - (-y)dy = 2y - \frac{y^3}{3} + \frac{y^2}{2}\Big|_{-1}^{2} = \boxed{4 - \frac{8}{3} + 2 - \left(-2 + 1 + \frac{1}{2}\right)}.$$

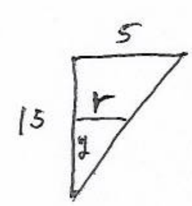
4. **Suppose that a conical tank is filled with oil which has a density of 50 lb/ft ^{3}. The radius at the top of the tank is 5 ft and the tank is 15 ft high. How much work is done in pumping the oil over the edge of the tank? Be sure to include units in your answer.**



Be sure to draw and label an axis. We have chosen the y-axis to point upward with $y = 0$ being the bottom of the tank. The work to lift the layer of oil with y-coordinate y is

Force $\cdot$ distance = weight $\cdot$ distance = volume $\cdot$ density $\cdot$ distance = $\pi r^2 t \cdot 50 \cdot$ distance,

where $t = dy$, r is computed from similar triangles $\frac{r}{y} = \frac{5}{15}$:



So, $r = \frac{1}{3}y$. The distance to lift the layer of oil whose y-coordinate is y is $15-y$. The work to lift one layer of oil is:

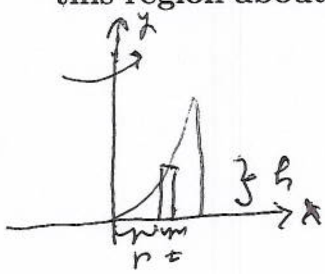
$$\pi(\tfrac{1}{3}y)^2 50(15-y)dy.$$

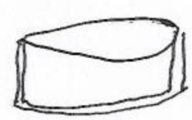
The work to lift all of the oil is

$$\frac{50\pi}{9}\int_0^{15}(15y^2-y^3)dy = \frac{50\pi}{9}\left(\frac{15y^3}{3}-\frac{y^4}{4}\right)\Big|_0^{15} = \frac{50(15)^4\pi}{9}\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$= \boxed{\frac{50(15)^4\pi}{9\cdot 12}\text{ft-lb.}}$$

5. **Consider the region bounded by $y = x^3$, $x = 1$, and $y = 0$. Revolve this region about the y-axis. Find the volume of the resulting solid.**





Spin the rectangle. Get a shell of volume $2\pi rht$, where $t = dx$, $r = x$, and $h = x^3$. The volume of the solid is

$$2\pi\int_0^1 x^4 dx = \frac{2\pi x^5}{5}\Big|_0^1 = \boxed{\frac{2\pi}{5}.}$$

6. **Find the limit of the sequence whose n^{th} term is $a_n = \left(\frac{3+n}{n}\right)^n$.**

The answer is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+3}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = \boxed{e^3}$$

because we know that $\lim_{n\to\infty}(1+\frac{r}{n})^n = e^r$.

7. Find $\int \frac{3x-4}{x^2-3x+2}dx$ **. Check your answer.**

Muliply both sides of

$$\frac{3x-4}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

by $(x-1)(x-2)$ to get

$$3x-4 = A(x-2) + B(x-1).$$

Plug in $x=1$ to learn $A=1$. Plug in $x=2$ to learn $B=2$. Check that

$$\frac{1}{x-1} + \frac{2}{x-2} = \frac{x-2+2x-2}{x^2-3x+2} = \frac{3x-4}{x^2-3x+2}. \checkmark.$$

Now we do the integral. The original problem is equal to

$$\int \frac{1}{x-1} + \frac{2}{x-2}dx = \boxed{\ln|x-1| + 2\ln|x-2| + C.}$$

8. Find $\int \frac{3x^2+8x+10}{x^3+2x^2+5x}dx$ **. Check your answer.**

Muliply both sides of

$$\frac{3x^2+8x+10}{x(x^2+2x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5}$$

by $x(x^2+2x+5)$ to see that:

$$3x^2+8x+10 = A(x^2+2x+5) + (Bx+C)x$$
$$3x^2+8x+10 = (A+B)x^2 + (2A+C)x + 5A$$

Eqauate the corresponding coefficients:

$$3 = A+B, \quad 8 = 2A+C, \quad 10 = 5A;$$

hence, $A=2$, $C=4$, and $B=1$. Check that

$$\frac{2}{x} + \frac{x+4}{x^2+2x+5} = \frac{2x^2+4x+10+x^2+4x}{x^3+2x^2+5x} = \frac{3x^2+8x+10}{x^3+2x^2+5x}.$$

Before we integrate, we see that

$$x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4$$

The problem is

$$\int \frac{2}{x} + \frac{x+4}{x^2+2x+5}dx = \int \frac{2}{x} + \frac{x+1}{(x+1)^2+4} + \frac{3}{(x+1)^2+4}dx$$

$$= \boxed{2\ln|x| + \frac{1}{2}\ln((x+1)^2+4) + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C}$$

9. **Find** $\int \ln x dx$ **. Check your answer.**

Use integration by parts. Let $u = \ln x$ and $dv = dx$. It follows that $du = \frac{dx}{x}$ and $v = x$. The original problem is $\int udv = uv - \int vdu$

$$= x\ln x - \int dx = \boxed{x\ln x - x + C.}$$

Check: The derivative of the proposed answer is

$$x\frac{1}{x} + \ln x - 1 = \ln x \checkmark.$$

10. **Find** $\int \frac{dx}{(1-x^2)^{3/2}}$ **. Check your answer.**

Let $x = \sin\theta$. It follows that $dx = \cos\theta d\theta$. It also follows that

$$(1-x^2) = 1 - \sin^2\theta = \cos^2\theta.$$

The integral is equal to

$$\int \frac{\cos\theta d\theta}{\cos^3\theta} = \int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta d\theta = \tan\theta + C = \frac{\sin\theta}{\cos\theta} + C = \boxed{\frac{x}{\sqrt{1-x^2}} + C.}$$

Check: The derivative of the proposed answer is

$$x(-1/2)(1-x^2)^{-3/2}(-2x) + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{x^2+1-x^2}{(1-x^2)^{3/2}} = \frac{1}{(1-x^2)^{3/2}}. \checkmark$$