Quiz 12, February 16, 2016

Find $\int \frac{1}{(x+1)(x^2+1)}dx$.

Answer: We use the technique of partial fractions and look for numbers A, B, and C with

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Multiply both sides by $(x+1)(x^2+1)$ and equate the corresponding coefficients to see that

$$\begin{aligned} 1 =& A(x^2+1) + (Bx+C)(x+1) \\ =& x^2(A+B) + x(B+C) + (A+C); \end{aligned}$$

hence,

$$\begin{aligned} 0 &= A+B \\ 0 &= B+C \\ 1 &= A+C \end{aligned}$$

Therefore, $A = -B$, $C = -B$, and $1 = -B-B$. Thus, $B = \frac{-1}{2}$ and $A = C = \frac{1}{2}$. We have shown that

$$\frac{1}{(x+1)(x^2+1)} = \frac{1}{2}\left[\frac{1}{x+1} + \frac{-x+1}{x^2+1}\right].$$

We verify the most recent equation before going any further. The right side is equal to

$$\frac{1}{2}\left[\frac{x^2+1+(1-x^2)}{(x+1)(x^2+1)}\right] = \frac{1}{(x+1)(x^2+1)},$$

as claimed. We now see that

$$\int \frac{1}{(x+1)(x^2+1)}dx = \frac{1}{2}\int \left[\frac{1}{x+1} + \frac{-x+1}{x^2+1}\right] dx =$$

$$\boxed{\frac{1}{2}\left[\ln|x+1| - \frac{1}{2}\ln(x^2+1) + \arctan x\right] + C}$$