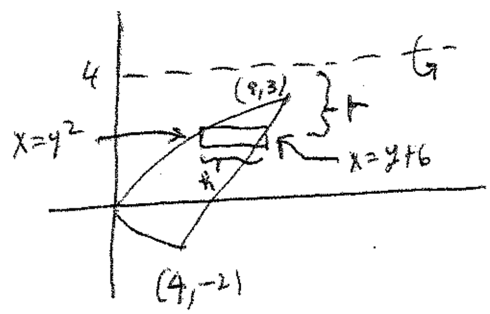
Math 142, Exam 4, Solutions Fall 2010

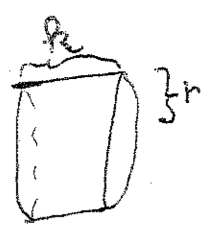
Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. *CIRCLE* your answer.
No Calculators or Cell phones.

1. **(6 points) Consider the region bounded by $x = y^2$ and $y = x - 6$. Revolve the region about $y = 4$. Find the volume of the resulting solid.**

We find the intersection points by solving $y = y^2 - 6$. This is $0 = y^2 - y - 6$, or $0 = (y-3)(y+2)$. So, $y = -2$ or $y = 3$. The intersection points are $(9,3)$ and $(4,-2)$. We draw the parabola and the line. The line $y = 4$ is a horizontal line above the region. The easiest way to do the problem is by using shells. We partition the y-axis from $y = -2$ to $y = 3$ and we express everything in terms of y. In particular the thickness of each shell is $t = dy$.



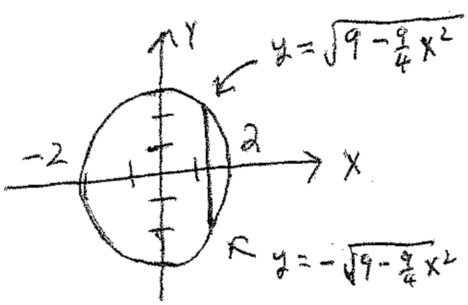


The radius of the shell with y-coordinate y is the big y-coordinate (namely 4) minus the little y-coordinate (namely y); so, $4 - y$. The height of the shell is the big x-coordinate minus the little x-coordinate; so, $(y+6) - y^2$. The volume of the shell with y-coordinate y is $2\pi rht = 2\pi(4-y)(y+6-y^2)dy$ The volume of the solid is

$$\boxed{2\pi \int_{-2}^{3} (4-y)(y+6-y^2)dy.}$$

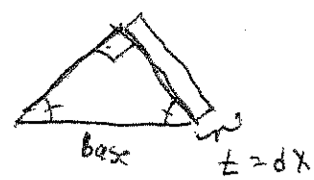
2. **(6 points) Consider a solid S. The base of S is an elliptical region with boundary curve $9x^2+4y^2=36$. The cross sections of S perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base. Find the volume of S.**

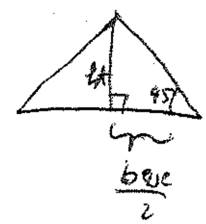
The boundary of the base is an ellipse. An ellipse is a slightly eccentric circle. We see that when $x=0$, then the ellipse passes through $y=3$ and $y=-3$. Also, when $y=0$, then $x=2$ and $x=-2$. We drew the base of S below. We slice S using slices that are parallel to the yz-plane. (That is, we chop the x-axis from $x=-2$ to $x=2$. We express everything in terms of x. So, for example, the thickness of the slice that we study is dx.) In particular, we focus on the slice of S that lives above the indicated line segment on the base of S. The slice of S above the line segment is drawn in the middle picture. This slice is a triangle with thickness dx. The volume of the slice is $\frac{1}{2}\cdot\text{base}\cdot\text{ht}\cdot\text{thickness}$. We draw one more picture to learn that the height of our slice is exactly equal to $\frac{1}{2}\cdot\text{base}$.



The base of S

The slice above the indicated line segment looks like





The slice with x-coordinate x has volume

$$\frac{1}{2}\cdot\text{base}\cdot\text{ht}\cdot\text{thickness}=\frac{1}{2}\cdot\text{base}\cdot\frac{1}{2}\cdot\text{base}\cdot dx=\frac{1}{4}\text{base}^2dx.$$

Of course, the base is the big y-coordinate minus the little y-coordinate or $2\sqrt{9-\frac{9}{4}x^2}$. The slice with x-coordinate x has volume

$$\frac{1}{4}\text{base}^2dx=\frac{1}{4}\left(2\sqrt{9-\frac{9}{4}x^2}\right)^2dx=\frac{1}{4}(4(9-\frac{9}{4}x^2))dx=(9-\frac{9}{4}x^2)dx.$$

The volume of the solid is

$$\int_{-2}^{2}(9-\frac{9}{4}x^2)dx=(9x-\frac{9}{4}\frac{x^3}{3})\Big|_{-2}^{2}=2(18-\frac{3\cdot 8}{4})=2(18-6)=\boxed{24}.$$

3. **(6 points) Consider the sequence $\{a_n\}$ with $a_0 = 0$, and for all $n \geq 1$, $a_n = \sqrt{6 + a_{n-1}}$. Prove that this sequence is increasing. Prove that this sequence is bounded. Deduce that the sequence converges. Find the limit of the sequence.**

We see that $0 < \sqrt{6} < \sqrt{6 + \sqrt{6}}$. So the series starts out to be increasing. Assume that after a while we have $a_{n-1} < a_n$. Then $6 + a_{n-1} < 6 + a_n$ and $\sqrt{6 + a_{n-1}} < \sqrt{6 + a_n}$. But $\sqrt{6 + a_{n-1}} = a_n$ and $\sqrt{6 + a_n} = a_{n+1}$. In other words, we have shown that: if a_{n-1} is less than a_n, then a_n is also less than a_{n+1}. So this sequence keeps on increasing forever. (This technique is called Mathematical Induction!)

We also use Mathematical Induction to show that 6 is an upper bound for this sequence. We have $0 < 6$, and $\sqrt{6} < 6$. Suppose $a_{n-1} < 6$, then $6 + a_{n-1} < 6 + 6$ and $\sqrt{6 + a_{n-1}} < \sqrt{12}$. But $\sqrt{6 + a_{n-1}}$ is also called a_n and $\sqrt{12} < 6$. We have shown that if a_{n-1} is less than 6, then a_n is also less than 6. So this sequence keeps on being less than 6 forever!

Our sequence is an increasing bounded sequence. The Completeness Axiom guarantees that our sequence has a limit. Let $L = \lim_{n\to\infty} a_n$. We may take $\lim_{n\to\infty}$ of both sides of $a_n = \sqrt{6 + a_{n-1}}$ to see that $\lim_{n\to\infty} a_n = \sqrt{6 + \lim_{n\to\infty} a_{n-1}}$; so, $L = \sqrt{6 + L}$ or $L^2 = 6 + L$. Thus, $L^2 - L - 6 = 0$ or $(L-3)(L+2) = 2$. So, $L = 3$ or $L = -2$ (But L can not be -2 because every element of the sequence is at least 0.) Thus, $\boxed{L = 3}$.

4. **(6 points) A ball is dropped from a height of 100 feet. Each time it hits the ground, it rebounds to $\frac{2}{3}$ its previous height. Find the total distance the ball travels before comming to rest.**

The most interesting part of the problem is that the first event is only down. Every other even is a round trip: up and down. The distance the ball travels is

$$100 + \tfrac{2}{3}(100) + \tfrac{2}{3}(100) + (\tfrac{2}{3})^2(100) + (\tfrac{2}{3})^2(100) + (\tfrac{2}{3})^3(100) + (\tfrac{2}{3})^3(100) + \dots$$

$$= 100 + \tfrac{2}{3}(200)(1 + \tfrac{2}{3} + (\tfrac{2}{3})^2 + (\tfrac{2}{3})^3 + \dots) = \boxed{100 + \tfrac{2}{3}(200)\tfrac{1}{1-\frac{2}{3}}}.$$

We used the fact that the geometric series with initial term a and radius r converges to $\frac{a}{1-r}$, provided $-1 < r < 1$.

5. **(6 points) Let $f(x)$ be the power series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2n+1}$. Where does $f(x)$ converge?**

It is clear that $f(3)$ converges. Fix some number x, with $x \neq 3$. We use the ratio test. Let

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{2(n+1)+1}}{\frac{(x-3)^n}{2n+1}} \right| = \lim_{n\to\infty} \frac{|x-3|^{n+1}(2n+1)}{|x-3|^n(2(n+1)+1)} = \lim_{n\to\infty} \frac{|x-3|(2n+1)}{(2n+3)}$$

$$= \lim_{n\to\infty} \frac{|x-3|(2+\frac{1}{n})}{(2+\frac{3}{n})} = |x-3|.$$

If $\rho < 1$, then $f(x)$ converges. If $1 < \rho$, then $f(x)$ diverges. If $\rho = 1$, then we need to do something else. Well, $\rho < 1$ for $|x-3| < 1$; that is, $-1 < x-3 < 1$ or $2 < x < 4$. (So if $2 < x < 4$, then $f(x)$ converges.) We see that $1 < \rho$ when $1 < |x-3|$. In other words, $1 < \rho$ when $1 < x-3$, that is: $4 < x$; also, $\rho < 1$ when $x - 3 < -1$, that is: $x < 2$. (So, if $x < 2$ or $4 < x$, then $f(x)$ converges.) We now study $f(x)$ for $x = 2$ and for $x = 4$. We see that $f(4) = \sum_{n=0}^{\infty} \frac{1}{2n+1}$. We compare $f(4)$ to the Harmonic series using the limit comparison test. We see that

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}.$$

This limit is a number. This limit is not zero or infinity. Thus, $f(4)$ and the Harmonic series both converge or both diverge. The Harmonic series diverges, thus, $f(4)$ diverges. We see that

$$f(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \dots .$$

This is an alternating series. The terms, in absolute value, decrease to zero. The Alternating Series Test applies. We conclude that $f(2)$ converges. Thus,

$f(x)$ converges for $2 \leq x < 4$ and $f(x)$ divergers everywhere else.

6. **(6 points) Find the second Taylor Polynomial $T_2(x)$ for $f(x) = \sqrt{x}$ about $a = 4$. Give an upper bound for the error that is introduced if $T_2(x)$ is used to approximate $f(x)$ for $4 \le x \le 4.2$.**

We compute

$$\begin{aligned} f(x) &= \sqrt{x} & f(4) &= 2 \\ f'(x) &= \tfrac{1}{2\sqrt{x}} & f'(4) &= \tfrac{1}{4} \\ f''(x) &= \tfrac{-1}{2\cdot 2(x)^{3/2}} & f''(4) &= \tfrac{-1}{32} \\ f'''(x) &= \tfrac{3}{2\cdot 2\cdot 2(x)^{5/2}}. \end{aligned}$$

We know that

$$T_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2.$$

In other words,

$$\boxed{T_2(x) = 2 + \frac{1}{4}(x-4) + \frac{-1}{2\cdot 32}(x-4)^2.}$$

Taylor's Theorem says that

$$|f(x) - T_2(x)| = |R_2(x)| = |\tfrac{f^3(c)}{3!}(x-4)^3|$$

for some c between x and 4. For us

$$|f^3(c)| = \frac{3}{8(c)^{5/2}}$$

and $4 \le c \le 4.2$. The biggest possible value of $f^3(c)$ occurs when c takes on the least value. (The fraction is largest when the demoninator is smallest.) So, $|f^3(c)| \le \frac{3}{8(4)^{5/2}} = \frac{3}{8\cdot 32}$. We have

$$|f(x) - T_2(x)| = |\tfrac{f^3(c)}{3!}(x-4)^3| \le \tfrac{3}{3!8\cdot 32}(.2)^3$$

for $4 \le x \le 4.2$ We conclude that

$T_2(x) = 2 + \frac{1}{4}(x-4) + \frac{-1}{2\cdot 32}(x-4)^2$ approximates $f(x) = \sqrt{4}$ with an error at most $\frac{3}{3!8\cdot 32}(.2)^3$ for $4 \le x \le 4.2$

7. **(7 points) Find $\lim\limits_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$.**

We see that

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) - x + \frac{1}{6}x^3}{x^5}$$

$$= \lim_{x\to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \dots\right) = \boxed{\frac{1}{5!}}.$$

8. **(7 points) Approximate $\int_0^1 x\cos(x^3)dx$ with an error at most 10^{-3}. Justify your answer very thoroughly.**

We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots .$$

It follows that

$$\cos x^3 = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots .$$

Thus,

$$\begin{aligned}\int_0^1 x\cos(x^3)dx &= \int_0^1 x(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots)dx \\ &= \int_0^1 (x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \dots)dx \\ &= (\tfrac{x^2}{2} - \tfrac{x^8}{2!\cdot 8} + \tfrac{x^{14}}{4!\cdot 14} - \tfrac{x^{20}}{6!\cdot 20} + \dots)\big|_0^1 \\ &= \frac{1}{2} - \frac{1}{2!\cdot 8} + \frac{1}{4!\cdot 14} - \frac{1}{6!\cdot 20} + \dots\end{aligned}$$

The above series is an alternating series. The terms, in absolute value, go to zero. The Alternating Series Test applies. We see that $\frac{1}{6!\cdot 20} < 10^{-3}$; so the alternating series test says that

$$|\int_0^1 x\cos(x^3)dx - (\tfrac{1}{2} - \tfrac{1}{2!\cdot 8} + \tfrac{1}{4!\cdot 14})| < \tfrac{1}{6!\cdot 20} < 10^{-3}.$$

Thus,

$$\boxed{\frac{1}{2} - \frac{1}{2!\cdot 8} + \frac{1}{4!\cdot 14} \text{ approximates } \int_0^1 x\cos(x^3)dx \text{ with an error at most } 10^{-3}.}$$