Math 142, Exam 2, Spring 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible.

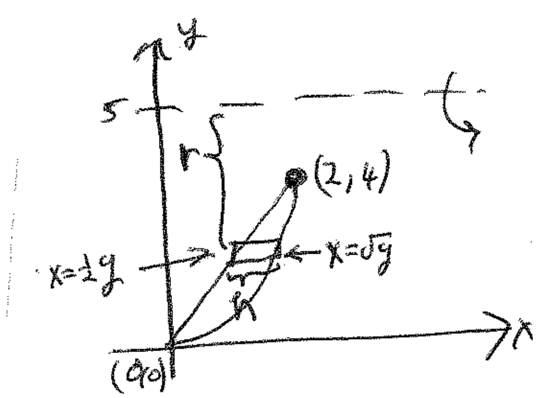
No Calculators or Cell phones.

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a, b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum_{i=1}^{n} f(x_i^*)\Delta_i$.

2. (6 points) **Consider the region bounded by $y = 2x$ and $y = x^2$. Revolve this region about the line $y = 5$. Find the volume of the resulting solid. You must draw a meaningful picture. What rectangle are you revolving? What does it become? Why are the dimensions what you say they are?**

The intersection points occur when $2x = x^2$; so, $0 = x(x-2)$ and $x = 0$ or $x = 2$. T The intersection points are $(0,0)$ and $(2,4)$. Consider the picture:



Chop the y-axis from $y = 0$ to $y = 4$. Associated to each piece of the y-axis, draw a rectangle. We focus on the rectangle with y-coordinate y. Rotate this rectangle get a shell of volume $2\pi rht$, where t is a little piece of the dy, $r = 5-y$, and $h = \sqrt{y} - y/2$. The volume of this one shell is $2\pi rht = 2\pi(5-y)(\sqrt{y}-y/2)dy$. The volume of the solid is

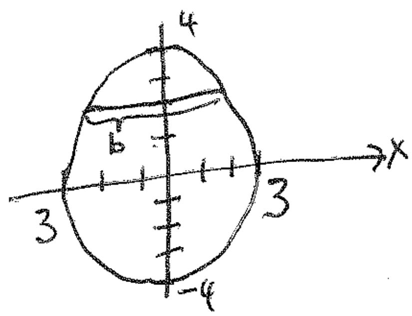
$$2\pi \int_0^4 (5\sqrt{y} - \tfrac{5}{2}y - y^{3/2} + \tfrac{y^2}{2})dy$$

$$= 2\pi \left(\tfrac{10}{3}y^{3/2} - \tfrac{5}{4}y^2 - \tfrac{2}{5}y^{5/2} + \tfrac{y^3}{6}\right)\Big|_0^4$$

$$= 2\pi\left(\tfrac{80}{3} - 20 - \tfrac{2}{5}(32) + \tfrac{64}{6}\right) = \boxed{\frac{2(68)\pi}{15}}.$$

3. (6 points) **Consider the solid whose base is bounded by $\frac{x^2}{9} + \frac{y^2}{16} = 1$ in the xy-plane. Each cross section of the solid perpendicular to the y-axis and perpendicular to the base is a square. Find the volume of the solid. You must draw a meaningful picture.**

The base of the solid is an ellipse:



Chop the y-axis from -4 to 4. Consider the slice of the solid with y-coordinate y. This slice has volume b^2t. We see that $\frac{b}{2}$ is the x-coordinate on the ellipse that corresponds to y; so, $\frac{b}{2} = 3\sqrt{1-\frac{y^2}{16}}$ and $b = 6\sqrt{1-\frac{y^2}{16}}$. The volume of the slice is

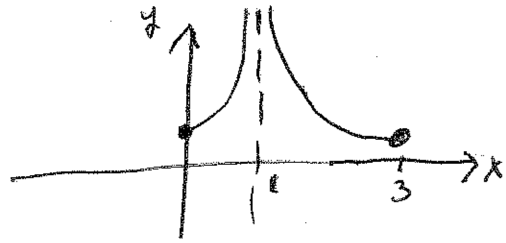
$$b^2t = 36(1-\tfrac{y^2}{16})dy.$$

The volume of the solid is

$$36\int_{-4}^{4}\left(1-\frac{y^2}{16}\right)dy = 36\left(y - \frac{y^3}{48}\right)\Big|_{-4}^{4} = \boxed{72(4-\tfrac{4}{3})}$$

4. (6 points) **Find** $\int_0^3 \frac{1}{(x-1)^2}dx$.

The function $f(x) = \frac{1}{(x-1)^2}$ goes to infinity as x gets near 1:



The answer will either be a positive number or $+\infty$. The integral is equal to

$$\lim_{b \to 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{a \to 1^+} \int_a^3 \frac{1}{(x-1)^2} dx$$

$$= \lim_{b \to 1^-} \frac{1}{-(x-1)}\bigg|_0^b + \lim_{a \to 1^+} \frac{1}{-(x-1)}\bigg|_a^3$$

$$= \lim_{b \to 1^-} \frac{1}{-(b-1)} - 1 + \lim_{a \to 1^+} \left(\frac{1}{-2} + \frac{1}{a-1} \right)$$

Both limits $\lim\limits_{b \to 1^-} \frac{1}{-(b-1)}$ and $\lim\limits_{a \to 1^+} \frac{1}{a-1}$ equal $+\infty$. The answer is $\boxed{+\infty}$.

5. (6 points) **Find** $\int \cos^5 x \sin^2 x dx$ **. You must check your answer.**

Save one $\cos x$. Let $u = \sin x$. It follows that $du = \cos x dx$. The integral is equal to

$$\int (1 - \sin^2 x)^2 \sin^2 x \cos x dx = \int (1-u^2)^2 u^2 du = \int (u^2 - 2u^4 + u^6) du$$

$$= \frac{u^3}{3} - \frac{2}{5}u^5 + \frac{u^7}{7} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{2}{5}\sin^5 x + \frac{\sin^7 x}{7} + C}.$$

Check. The derivative of the proposed answer is

$$\sin^2 x \cos x - 2\sin^4 x \cos x + \sin^6 x \cos x = \cos x \sin^2 x (1 - 2\sin^2 x + \sin^4 x)$$

$$= \cos x \sin^2 x (1 - \sin^2 x)^2.$$

6. (5 points) **Find** $\int \frac{e^x}{1+e^{2x}} dx$ **. You must check your answer.**

Let $u = e^x$. It follows that $du = e^x dx$. The integral is equal to

$$\int \frac{du}{1+u^2} = \arctan u + C = \boxed{\arctan(e^x) + C}.$$

<u>Check</u> . The derivative of the proposed answer is

$$\frac{e^x}{1+e^{2x}}. \checkmark$$

7. (5 points) **Find** $\int \frac{1}{x^2+4x+5} dx$ **. You must check your answer.**

The problem is equal to $\int \frac{1}{(x+2)^2+1} dx$. Let $u = x + 2$. It follows that $du = dx$. The integral is equal to

$$\int \frac{du}{1+u^2} = \arctan u + C = \boxed{\arctan(x+2) + C}.$$

<u>Check</u> . The derivative of the proposed answer is

$$\frac{1}{1+(x+2)^2}. \checkmark$$

8. (5 points) **Find** $\int \sec^3 x \tan^5 x dx$ **. You must check your answer.**

Save one $\sec x \tan x$. Convert the rest of the $\tan x$'s to $\sec x$. Let $u = \sec x$. It follows that $du = \sec x \tan x dx$. The original integral is

$$\int u^2(u^2-1)^2 du = \int u^6 - 2u^4 + u^2 du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$$

$$= \boxed{\frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C}.$$

<u>Check</u> . The derivative of the proposed answer is

$$\sec^6 x \sec x \tan x - 2\sec^4 x \sec x \tan x + \sec^2 x \sec x \tan x$$

$$= \sec^3 x \tan^x(\sec^4 x - 2\sec^2 x + 1) = \sec^3 x \tan x(\sec^2 x - 1)^2$$

9. (5 points) Find $\int \frac{x-1}{x^4+4x^2}dx$. You must check your answer.

We see that $x^4+4x^2 = x^2(x^2+4)$. Write

$$\frac{x-1}{x^4+4x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}.$$

Multiply both sides by x^4+4x^2 to get

$$x-1 = Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2;$$

so,

$$x-1 = (A+C)x^3 + (B+D)x^2 + 4Ax + 4B.$$

Equate the corresponding coefficients to obtain $B = -\frac{1}{4}$, $A = \frac{1}{4}$, $B+D=0$ (hence $D=\frac{1}{4}$), and $A+C=0$ (so $C=-\frac{1}{4}$). We check that

$$\frac{1}{4}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{-x+1}{x^2+4}\right] = \frac{1}{4}\left[\frac{x(x^2+4)-(x^2+4)+(-x+1)x^2}{x^2(x^2+4)}\right] = \frac{x-1}{x^2(x^2+4)}.$$

The original integral is equal to

$$\frac{1}{4}\int\left(\frac{1}{x} - \frac{1}{x^2} + \frac{-x+1}{x^2+4}\right)dx = \boxed{\frac{1}{4}\left[\ln|x| + \frac{1}{x} - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\arctan\frac{x}{2}\right] + C}$$