Math 142, Final Exam, Fall 2009

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 200 points. There are **20** problems. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Find $\int 3\sqrt{4+2x}dx$. Check your answer.**

Let $u = 4+2x$; so, $du = 2dx$. The integral is

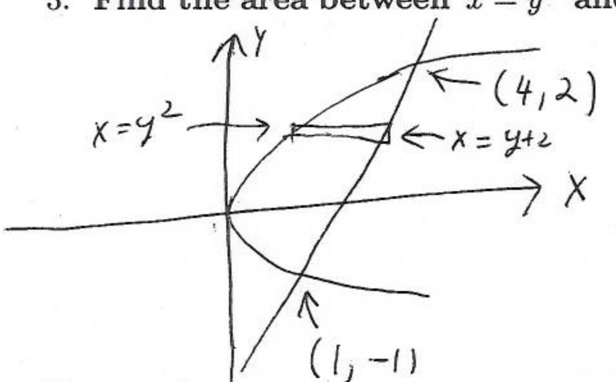
$$(3/2)\int u^{1/2}du = (3/2)(2/3)u^{3/2} + C = \boxed{(4+2x)^{3/2}+C}.$$

Check. The derivative of the proposed answer is $(3/2)(4+2x)^{1/2}2$ ✓ .

2. **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a,b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum_{i=1}^{n} f(x_i^*)\Delta_i$.

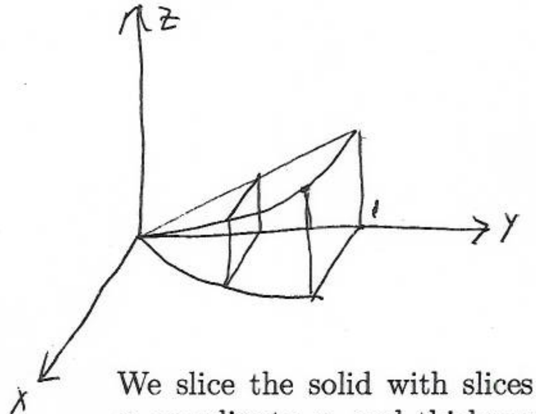
3. **Find the area between $x = y^2$ and $y = x - 2$.**

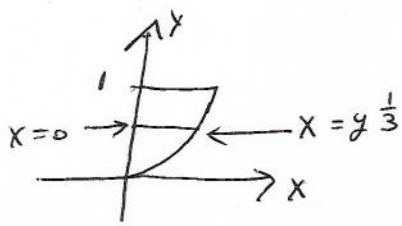


The area is

$$\int_{-1}^{2}(y+2-y^2)dy = \left.\frac{y^2}{2} + 2y - \frac{y^3}{3}\right|_{-1}^{2} = 2+4-\frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \boxed{\frac{9}{2}}.$$

4. **Find the volume of the solid whose base in the xy plane is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.**

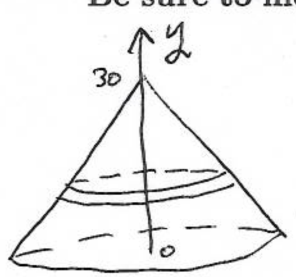




We slice the solid with slices parallel to the xz-plane. The indicated slice has y-coordinate y and thickness dy. The volume of the indicated slice is $y^{2/3}dy$. The volume of the solid is

$$\int_0^1 y^{2/3}dy = \left.\frac{3y^{5/3}}{5}\right|_0^1 = \boxed{\frac{3}{5}}.$$

5. **A conical water tank whose base is a circle of radius 10 feet and whose height is 30 ft is filled with water to a depth of 15 feet. How much work is required to pump all of the water out through a hole in the top of the tank? (The density of water is 62.4 pounds per cubic foot.) Be sure to include units in your answer.**



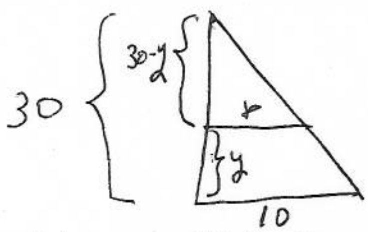
We find the work required to lift the layer of water whose y-coordinate is y and whose thickness is dy. The work to lift this layer is

the weight of the layer times the distance it must be lifted

$= 62.4$ times the volume of the layer times the distance it must be lifted

$$= 62.4\pi r^2 d\, dy,$$

where r is the radius of the layer and d is the distance the layer we lift the layer.



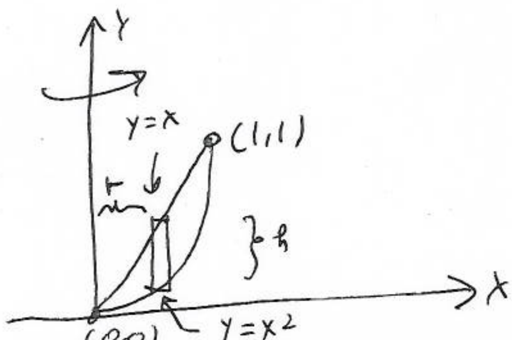
The distance we lift the layer of water is $d = 30 - y$. Similar triangles shows us that $\frac{r}{30-y} = \frac{10}{30}$; so, $r = \frac{1}{3}(30-y)$. The work to lift this layer is

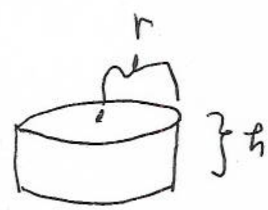
$$= 62.4\pi \left(\frac{1}{3}(30-y)\right)^2 (30-y)\, dy.$$

The work to lift all of the water out of the tank is

$$\frac{62.4\pi}{9}\int_0^{15} (30-y)^3\, dy = \frac{62.4\pi}{9}\frac{(30-y)^4}{-4}\bigg|_0^{15} = \boxed{\frac{62.4\pi}{36}(30^4 - 15^4)\text{ft-lbs.}}$$

6. Consider the region in the first quadrant bounded by $y = x$ and $y = x^2$. Rotate this region about the y-axis. Find the volume.





Spin the rectangle get a shell of volume $2\pi rht = 2\pi x(x - x^2)dx$. The volume of the solid is

$$2\pi\int_0^1 x(x - x^2)dx = 2\pi\int_0^1 (x^2 - x^3)dx = 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\bigg|_0^1 = \frac{2\pi}{12} = \boxed{\frac{\pi}{6}}.$$

7. **Find the length of** $y = x^{3/2}$ **from** $(1,1)$ **to** $(2, 2\sqrt{2})$.

The arc length is

$$\int_1^2 \sqrt{1+(dy/dx)^2}dx = \int_1^2 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_1^2 \sqrt{1+(9/4)x}dx$$

$$= \frac{4(2)(1+(9/4)x)^{3/2}}{9(3)}\Bigg|_1^2 = \boxed{\frac{8}{27}\left(\left(1+\frac{9}{2}\right)^{3/2} - \left(1+\frac{9}{4}\right)^{3/2}\right)}$$

8. **Find** $\int xe^x dx$. **Check your answer.**

Try integration by parts. Let $u = x$ and $dv = e^x dx$. Calculate $du = dx$ and $v = e^x$. The given problem is

$$\int udv = uv - \int vdu = xe^x - \int e^x dx = \boxed{xe^x - e^x + C.}$$

Check. The derivative of the proposed answer is

$$xe^x + e^x - e^x \checkmark.$$

9. **Find** $\int \tan^2 x \sec^4 x dx$. **Check your answer.**

The integral is

$$\int \tan^2 x \sec^2 x \sec^2 x dx = \int \tan^2 x(\tan^2 x + 1)\sec^2 x dx.$$

Let $u = \tan x$. It follows that $du = \sec^2 x dx$. The integral is

$$\int (u^4 + u^2)du = \frac{u^5}{5} + \frac{u^3}{3} + C = \boxed{\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C}.$$

Check. The derivative of the proposed answer is

$$\tan^4 x \sec^2 x + \tan^2 x \sec^2 x = \tan^2 x \sec^2 x(\tan^2 x + 1) = \tan^2 x \sec^4 x \checkmark.$$

10. **Find** $\int \frac{\sqrt{x^2-25}}{x}dx$ **. Check your answer.**

Let $x = 5\sec\theta$. It follows that $x^2-25 = 25\sec^2\theta - 25 = 25(\sec^2\theta - 1) = 25\tan^2\theta$. It also follows that $dx = 5\sec\theta\tan\theta d\theta$. The original integral is equal to

$$\int \frac{5\tan\theta(5\sec\theta\tan\theta d\theta)}{5\sec\theta} = 5\int \tan^2\theta d\theta = 5\int (\sec^2\theta - 1)d\theta = 5(\tan\theta - \theta) + C$$

$$= 5\left(\frac{\sqrt{x^2-25}}{5} - \operatorname{arcsec}(x/5)\right) + C$$

$$= \boxed{\sqrt{x^2-25} - 5\operatorname{arcsec}(x/5) + C.}$$

Check. The derivative of the proposed answer is

$$\frac{2x}{2\sqrt{x^2-25}} - \frac{(\frac{1}{5})5}{\frac{x}{5}\sqrt{\frac{x^2}{25}-1}} = \frac{x^2}{x\sqrt{x^2-25}} - \frac{25(\frac{1}{5})5}{25(\frac{x}{5})\sqrt{\frac{x^2}{25}-1}}$$

$$= \frac{x^2}{x\sqrt{x^2-25}} - \frac{25}{x\sqrt{x^2-25}} = \frac{\sqrt{x^2-25}}{x} \checkmark.$$

11. **Find** $\int \frac{x+1}{(x-1)^2}dx$ **. Check your answer.**

We solve

$$\frac{x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

Multiply both sides by $(x-1)^2$ to obtain

$$x + 1 = A(x-1) + B.$$

Equate the corresponding coefficients:

$$1 = A \quad \text{and} \quad 1 = B - A.$$

So, $A = 1$ and $B = 2$. We check this before we go any further:

$$\frac{1}{x-1} + \frac{2}{(x-1)^2} = \frac{x-1+2}{(x-1)^2} = \frac{x+1}{(x-1)^2}. \checkmark$$

So the original problem is the same as

$$\int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2}\right) dx = \boxed{\ln|x-1| - \frac{2}{x-1} + C.}$$

12. Find $\int_0^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$.

The integral is equal to

$$\lim_{b\to\infty} \int_0^b \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{b\to\infty} -2e^{-\sqrt{x}}\Big|_0^b = \lim_{b\to\infty} -2e^{-\sqrt{b}} + 2 = \lim_{b\to\infty} \frac{-2}{e^{\sqrt{b}}} + 2 = \boxed{2.}$$

13. Find the limit of the sequence whose n^{th} term is $a_n = \dfrac{\sin(4n)}{n}$.

L'hopital's rule has nothing to do with this problem. We have

$$0 = \lim_{n\to\infty} \frac{-1}{n} \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{1}{n} = 0. \text{ Thus } \boxed{\lim_{n\to\infty} a_n = 0}.$$

14. Consider the series $\sum\limits_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. Find a closed formula for the n^{th} partial sum $s_n = \sum\limits_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. Be sure to answer the question I have asked!

We compute a partial sum of a telescoping series:

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(\tfrac{1}{1} - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{n-2} - \tfrac{1}{n-1}\right) + \left(\tfrac{1}{n-1} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right)$$

$$= \boxed{1 - \frac{1}{n+1}.}$$

15. Consider the series $\sum\limits_{k=1}^{\infty} (23)^k$. Find a closed formula for the n^{th} partial sum $s_n = \sum\limits_{k=1}^{n} (23)^k$. Be sure to answer the question I have asked!

We compute a partial sum of a geometric series:

$$22s_n = 23s_n - s_n = 23\sum_{k=1}^{n} (23)^k - \sum_{k=1}^{n} (23)^k$$

$$= (23)^2 + \cdots + (23)^{n+1} - ((23) + \ldots (23)^n) = (23)^{n+1} - 23;$$

so,

$$\boxed{s_n = \frac{(23)^{n+1} - 23}{22}}.$$

16. **Does the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}+1}{k^2+2k}$ converge? Justify your answer.**

We apply the limit comparison test. We compare the given series to $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$. We calculate

$$\lim_{k\to\infty} \frac{\frac{\sqrt{k}+1}{k^2+2k}}{\frac{1}{k^{3/2}}} = \lim_{k\to\infty} \frac{k^2+k^{3/2}}{k^2+2k} = \lim_{k\to\infty} \frac{1+\frac{1}{k^{1/2}}}{1+\frac{2}{k}} = 1.$$

Since 1 is a number and 1 is not zero or infinity, we may apply the limit comparison test. The series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is the p-series with $p = 3/2 > 1$. This series converges; therefore,

$$\boxed{\sum_{k=1}^{\infty} \frac{\sqrt{k}+1}{k^2+2k} \text{ also converges.}}$$

17. **Find the third Taylor polynomial for $f(x) = \sqrt{x}$ about $c = 1$.**

We compute

$$\begin{array}{ll} f(x) = \sqrt{x} & f(1) = 1 \\ f'(x) = \frac{1}{2}x^{-1/2} & f'(1) = \frac{1}{2} \\ f''(x) = \frac{-1}{4}x^{-3/2} & f''(1) = \frac{-1}{4} \\ f'''(x) = \frac{3}{8}x^{-5/2} & f''(1) = \frac{3}{8} \end{array}$$

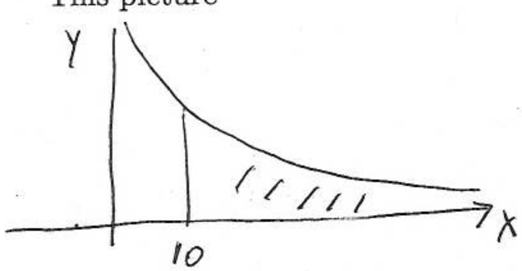
So,

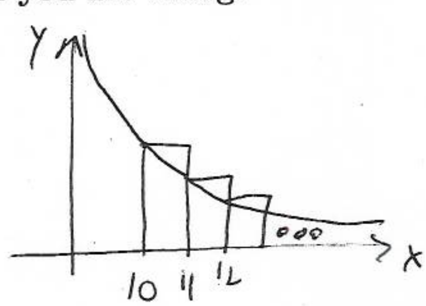
$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= \boxed{1 + \frac{1}{2}(x-1) + \frac{-1}{8}(x-1)^2 + \frac{3}{(8)(3!)}(x-1)^3.}$$

18. **Approximate $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n^2}$. I want you to give both an under estimate and an over estimate. Explain what you are doing.**

This picture

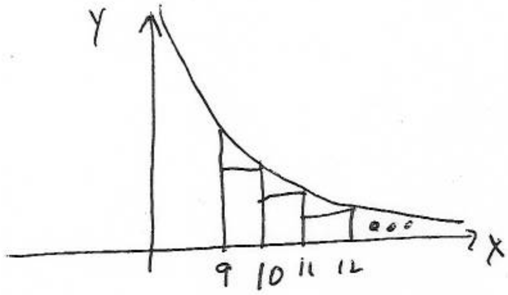


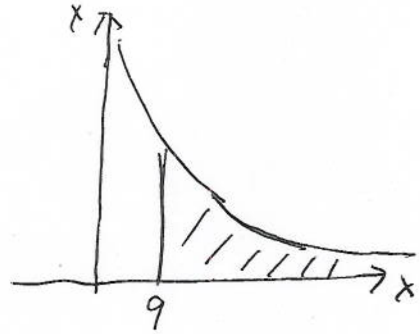


shows that

$$\int_{10}^{\infty} \frac{1}{x^2} dx \le \sum_{n=10}^{\infty} \frac{1}{n^2}.$$

This picture





shows that

$$\sum_{n=10}^{\infty} \frac{1}{n^2} \le \int_{9}^{\infty} \frac{1}{x^2} dx$$

We compute

$$\int_{10}^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_{10}^{b} = \lim_{b\to\infty} \frac{-1}{b} + \frac{1}{10} = \frac{1}{10}.$$

We compute

$$\int_{9}^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_{9}^{b} = \lim_{b\to\infty} \frac{-1}{b} + \frac{1}{9} = \frac{1}{9}.$$

We conclude that

$$\boxed{\frac{1}{10} \le \sum_{n=10}^{\infty} \frac{1}{n^2} \le \frac{1}{9}.}$$

19. **Approximate $e^{\frac{-1}{100}}$ with an error at most 10^{-9}. Explain what you are doing**

We know

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots ,$$

for all x. So, in particular,

$$e^{\frac{-1}{100}} = 1 - \frac{1}{100} + \frac{1}{2(100)^2} - \frac{1}{3!(100)^3} + \frac{1}{4!(100)^4} - \frac{1}{5!(100)^5} + \dots .$$

The alternating series test applies to the above series because the series alternates and the absolute value of the terms is decreasing to zero. We see that $\frac{1}{4!(100)^4} < (10)^{-9}$. We conclude that

$$\boxed{1 - \frac{1}{100} + \frac{1}{2(100)^2} - \frac{1}{3!(100)^3} \text{ approximates } e^{\frac{-1}{100}} \text{ with an error at most } 10^{-9}.}$$

20. What is the exact value of the sum $\sum_{k=1}^{\infty} \frac{1}{k(4^k)} = \frac{1}{4} + \frac{1}{2(4^2)} + \frac{1}{3(4^3)} + \frac{1}{4(4^4)} \cdots$?

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,$$

for all x with $-1 < x < 1$. Integrate to see that

$$-\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots,$$

for all x with $-1 < x < 1$. Plug in $x = 0$ to see that $C = 0$. So,

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots,$$

for $-1 < x < 1$. Plug in $x = \frac{1}{4}$ to learn

$$\boxed{-\ln \tfrac{3}{4}} = \tfrac{1}{4} + \tfrac{1}{2(4^2)} + \tfrac{1}{3(4^3)} + \tfrac{1}{4(4^4)} \cdots.$$