Quiz 29, April 14, 2016

Find a polynomial which approximates $F(x) = \int_0^x \arctan(t)dt$ with an error at most $\frac{1}{1000}$ when $0 \le x \le \frac{1}{2}$.

Answer: We know $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \dots$ for $-1 < y < 1$. Replace y with $-t^2$. Notice that $-1 < -t^2 < 1$ precisely when $-1 < t < 1$; hence,

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \dots \quad \text{for } -1 < t < 1.$$

Integrate to learn that

$$\arctan(t) = C + t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots, \quad \text{for } -1 < t < 1.$$

Plug in $t = 0$ to see that $0 = C + 0$; hence,

$$\arctan(t) = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots, \quad \text{for } -1 < t < 1.$$

Now we are ready to work on the problem,

$$F(x) = \int_0^x \arctan(t)dt = \int_0^x \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots\right)dt$$

$$= \left(\frac{t^2}{2} - \frac{t^4}{3\cdot 4} + \frac{t^6}{5\cdot 6} - \frac{t^8}{7\cdot 8} + \dots\right)\Bigg|_0^x$$

$$\left(\frac{x^2}{2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \dots\right)$$

We only consider x with $0 \le x \le \frac{1}{2}$. For such x, the Alternating Series Test applies because the series alternates,

$$\frac{x^2}{2} > \frac{x^4}{3\cdot 4} > \frac{x^6}{5\cdot 6} > \cdots,$$

and $\lim_{n\to\infty} \frac{x^n}{(n-1)n} = 0$. We would like to pick an even number n with

$$\frac{x^{n+2}}{(n+1)\cdot(n+2)} < \frac{1}{1000}$$

for $0 \le x \le 1/2$. So we want

$$\frac{1}{2^{n+2}(n+1)\cdot(n+2)} < \frac{1}{1000}.$$

We want

$$1000 \le 2^{n+2}(n+1)(n+2).$$

We see that if $n = 6$, then

$$1000 < 2^8(7)(8) = 2^{n+2}(n+1)(n+2).$$

We conclude that

$$\boxed{\frac{x^2}{2} - \frac{x^4}{3 \cdot 4} + \frac{x^6}{5 \cdot 6} \text{ approximates } F(x) \text{ with an error at most } \frac{1}{1000}, \text{ when } 0 \leq x \leq 1/2.}$$