Quiz 27, March 31, 2016

Consider the series $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + (\frac{-1}{2})^n(x-3)^n + \dots$.

(a) Where does the series converge?

(b) What is the sum of the series?

(c) Integrate the series term-by-term. What is the sum of the new series?

Answer: We see that $f(x)$ is the geometric series with initial term 1 and ratio $r = \frac{-(x-3)}{2}$. Thus, $f(x)$ converges when $-1 < \frac{-(x-3)}{2} < 1$, or $-2 < -(x-3) < 2$, or $-2 < x-3 < 2^{\dagger}$, or $1 < x < 5$.

$$\boxed{\text{(a)} \quad f(x) \text{ converges for } 1 < x < 5.}$$

If $1 < x < 5$, then $f(x) = \frac{a}{1-r} = \frac{1}{1-\frac{-(x-3)}{2}} = \frac{2}{2+(x-3)} = \frac{2}{x-1}$

$$\boxed{\text{(b)} \quad \text{If } 1 < x < 5, \text{ then } \frac{2}{x-1} = f(x).}$$

Integrate both sides of the answer to (b) to learn that if $1 < x < 5$, then

$$\ln(x-1) + C = x - \frac{1}{2\cdot 2}(x-3)^2 + \frac{1}{4\cdot 3}(x-3)^3 + \cdots + \frac{(-1)^n}{2^n\cdot(n+1)}(x-3)^{n+1} + \dots .$$

Plug in $x = 3$ to learn

$$\ln(2) + C = 3.$$

So $C = 3 - \ln 2$ and

$$\ln(x-1) + 3 - \ln 2 = x - \frac{1}{2\cdot 2}(x-3)^3 + \frac{1}{4\cdot 3}(x-3)^2 + \cdots + \frac{(-1)^n}{2^n\cdot(n+1)}(x-3)^{n+1} + \dots .$$

It will look prettier if we move the 3 to the other side:

(c) If $1 < x < 5$, then
$\ln(x-1) - \ln 2 = (x-3) - \frac{1}{2\cdot 2}(x-3)^2 + \frac{1}{4\cdot 3}(x-3)^3 + \cdots + \frac{(-1)^n}{2^n\cdot(n+1)}(x-3)^{n+1} + \dots$.

† This one is tricky. In general, if $a < b < c$, then $-c < -b < -a$. For us, $-2 < -(x-3) < 2$; so $-2 < -(-(x-3)) < -(-2)$ and this is $-2 < x-3 < 2$.