Quiz 22, November 15, 2016

Let $f(x) = \sum_{n=1}^{\infty} \frac{4^n x^{2n}}{n}$. For which x does $f(x)$ converge? For which x does $f(x)$ diverge? Explain thoroughly. Write complete thoughts.

Answer: Observe that $f(0)$ converges. Fix a number x with $x \neq 0$. We apply the ratio test. Let

$$\rho = \lim_{n\to\infty} \frac{\frac{4^{n+1}|x|^{2(n+1)}}{n+1}}{\frac{4^n|x|^{2n}}{n}} = \lim_{n\to\infty} \frac{4^{n+1}|x|^{2(n+1)}}{n+1}\frac{n}{4^n|x|^{2n}} = \lim_{n\to\infty} 4|x|^2\frac{n}{n+1} = \lim_{n\to\infty} 4|x|^2\frac{1}{1+\frac{1}{n}} = 4|x|^2.$$

If $\rho < 1$, then $f(x)$ converges. If $1 < \rho$, then $f(x)$ diverges. We will need to use a different test when $\rho = 1$.

We see that $\rho < 1$ when $4|x|^2 < 1$. The most recent inequality is equivalent to $x^2 < \frac{1}{4}$ and this is equivalent to $-\frac{1}{2} < x < \frac{1}{2}$.

We see that $1 < \rho$, when $1 < 4x^2$. The most recent inequality is equivalent to $\frac{1}{4} < x^2$ (or $x^2 < -\frac{1}{4}$ but a perfect square is never negative). Thus $1 < \rho$ when $\frac{1}{2} < x$ or when $x < -\frac{1}{2}$.

We study the end points seperately.

We see that

$$f\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{4^n(\frac{1}{2})^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series which diverges.

We see that

$$f\left(-\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{4^n(-\frac{1}{2})^{2n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is also the harmonic series which still diverges.

Thus, $\boxed{f(x) \text{ converges for } -\frac{1}{2} < x < \frac{1}{2} \text{ and diverges everywhere else}}$.