Quizzes 21 and 22, March 22, 2016

Question 1. Does the series $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ converge? Justify your answer.

Answer: Compare the given series to $\sum_{n=1}^{\infty} \frac{1}{e^n}$. Notice that $\frac{e^n}{1+e^{2n}}$ is positive and

$$\frac{e^n}{1+e^{2n}} < \frac{1}{e^n}$$

because $e^{2n} < 1+e^{2n}$. The series $\sum_{n=1}^{\infty} \frac{1}{e^n}$ is the geometric series with initial term $a = \frac{1}{e}$ and ratio $r = \frac{1}{e}$. We see that $-1 < \frac{1}{e} < 1$; so the geometric series $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges. It follows from the first part of the comparison test that

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}} \text{ also converges.}}$$

Question 2. Consider the sequence $a_1 = 2$, $a_2 = 2+\frac{1}{2}$, $a_3 = 2+\frac{1}{2+\frac{1}{2}}$, $a_4 = 2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}$, $\dots$. Assume the the sequence converges. Find the limit of the sequence.

Answer: This sequence is given recursively, $a_{n+1} = 2+\frac{1}{a_n}$. We are told that the sequence converges. Let $L = \lim_{n\to\infty} a_n$. Take $\lim_{n\to\infty}$ of both sides of $a_{n+1} = 2+\frac{1}{a_n}$ to obtain $L = 2+\frac{1}{L}$. Multiply both sides by L: $L^2 = 2L+1$ or $L^2-2L-1 = 0$. Use the quadratic formula to see that $L = \frac{2\pm\sqrt{4+4}}{2}$; so $L = \frac{2\pm 2\sqrt{2}}{2} = 1\pm\sqrt{2}$. We see that $1-\sqrt{2} < 2 < 1+\sqrt{2}$. Every term in our sequence is 2 plus something positive; so every term in our sequence is more than 2. The limit of our sequence is at least 2. The limit of our sequence is not $1-\sqrt{2}$. The limit of our sequence must equal $\boxed{1+\sqrt{2}}$.