Quiz 18, October 26, 2016

Approximate $\sum_{k=1}^{\infty} \frac{1}{k^3}$ with an error at most $\frac{1}{100}$. Explain what you are doing. Write in complete sentences.

Answer: Look at the picture. We approximate the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k^3}$ with the finite sum $\sum_{k=1}^{N} \frac{1}{k^3}$. We will pick N large enough so that the error that is incurred when this approximation is made is at most $\frac{1}{100}$. We see that the error is equal to

$$\left|\sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{N} \frac{1}{k^3}\right| = \sum_{k=N+1}^{\infty} \frac{1}{k^3}$$

$$= \text{the area inside the boxes} < \text{the area under the curve}$$

$$= \int_N^{\infty} \frac{1}{x^3} dx = \lim_{b\to\infty} \left.\frac{-1}{2x^2}\right|_N^b = \lim_{b\to\infty} \left(\frac{-1}{2b^2} + \frac{1}{2N^2}\right) = \frac{1}{2N^2}.$$

We want the error to be at most $\frac{1}{100}$. We have shown that the error is at most $\frac{1}{2N^2}$; so, we make

$$\frac{1}{2N^2} \leq \frac{1}{100}.$$

The most recent inequality is equivalent to $50 \leq N^2$ and this is equivalent to $8 \leq N$. We conclude that

$$\boxed{\sum_{k=1}^{8} \frac{1}{k^3} \text{ approximates } \sum_{k=1}^{\infty} \frac{1}{k^3} \text{ with an error at most } \frac{1}{100}}.$$