Quiz 11, September 22, 2016

Find $\int \frac{dx}{(x^2-1)^2}dx$.

Answer: We use the technique of partial fractions and look for numbers A, B, C, and D with

$$\frac{1}{(x^2-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

Multiply both sides by $(x^2-1)^2$ and equate the corresponding coefficients to see that

$$\begin{aligned}
1 =& A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2 \\
=& A(x^2-1)(x+1) + B(x^2+2x+1) + C(x^2-1)(x-1) + D(x^2-2x+1) \\
=& A(x^3+x^2-x-1) + B(x^2+2x+1) + C(x^3-x^2-x+1) + D(x^2-2x+1) \\
=& x^3(A+C) + x^2(A+B-C+D) + x(-A+2B-C-2D) + (-A+B+C+D)
\end{aligned}$$

hence,

$$\begin{aligned}
0 &= A+C \\
0 &= A+B-C+D \\
0 &= -A+2B-C-2D \\
1 &= -A+B+C+D.
\end{aligned}$$

It follows that

$$\begin{aligned}
A &= -C \\
0 &= B-2C+D \\
B &= D \\
1 &= 2C+2D.
\end{aligned}$$

The second equation now yields $C = D$; hence, the fourth equation becomess $1 = 4D$. We conclude that

$$D = C = B = \tfrac{1}{4}, \quad \text{and} \quad A = -\tfrac{1}{4}.$$

We have shown that

$$\frac{1}{(x^2-1)^2} = \frac{1}{4}\left[\frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x+1} + \frac{1}{(x+1)^2}\right].$$

We verify the most recent equation before going any further. The right side is equal to

$$\frac{1}{4}\left[\frac{-x+2}{(x-1)^2} + \frac{x+2}{(x+1)^2}\right] = \frac{1}{4}\left[\frac{(-x+2)(x^2+2x+1)+(x+2)(x^2-2x+1)}{(x^2-1)^2}\right]$$

$$= \frac{1}{4}\left[\frac{(-x^3-2x^2-x)+(2x^2+4x+2)+(x^3-2x^2+x)+(2x^2-4x+2)}{(x^2-1)^2}\right] = \frac{1}{4}\left[\frac{4}{(x^2-1)^2}\right]$$

$$= \frac{1}{(x^2-1)^2},$$

as claimed. We now see that

$$\int \frac{1}{(x^2-1)^2} dx = \frac{1}{4} \int \left[\frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x+1} + \frac{1}{(x+1)^2} \right] dx$$

$$= \boxed{\frac{1}{4} \left[-\ln|x-1| - \frac{1}{(x-1)} + \ln|x+1| - \frac{1}{(x+1)} \right] + C}.$$