Math 142 Final Exam Fall 2004 Solutions

PRINT Your Name:____________________________________

There are 20 problems on 10 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your grade from VIP.

I will post the solutions on my website when the exam is finished.

1. **Find** $\frac{d}{dx}\left(xe^{x^2}\right)$**.**

$$\boxed{x(2xe^{x^2}) + e^{x^2}.}$$

2. **Find** $\int xe^{x^2}\,dx$**. Check your answer.**

Let $u = x^2$. Then $du = 2x\,dx$; so the integral is $\int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C = \boxed{\frac{e^{x^2}}{2} + C}$.

3. **Simplify** $\sin(2\arccos(\frac{3}{4}))$**.**

The expression is

$$2\sin(\arccos(\tfrac{3}{4}))\cos(\arccos(\tfrac{3}{4})) = \boxed{2\tfrac{\sqrt{7}}{4}\tfrac{3}{4}.}$$

Draw a right triangle with the adjacent equal to 3, hypotenuse equal to 4, and opposite equal to $\sqrt{7}$ to see that $\sin(\arccos(\frac{3}{4})) = \frac{\sqrt{7}}{4}$.

4. **Find** $\int \sin^4 x\cos^3 x\,dx$**. Check your answer.**

Save one $\cos x$ turn the other two $\cos x$'s into $\sin x$'s. Then let $u = \sin x$. It follows that $du = \cos x$. The integral is

$$\int \sin^4 x(1-\sin^2 x)\cos x\,dx = \int u^4(1-u^2)\,du = \int (u^4 - u^6)\,du = \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \boxed{\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.}$$

Check. The derivative of the proposed answer is

$$\sin^4 x\cos x - \sin^6 x\cos x = \sin^4 x\cos x(1-\sin^2 x) = \sin^4 x\cos x\cos^2 x. \checkmark$$

5. **Find** $\int \sin^2 x\, dx$ **.**

The integral is

$$\frac{1}{2}\int 1 - \cos 2x\, dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C.$$

6. **Find** $\int \frac{x}{x^2+4}\, dx$ **. Check your answer.**

Let $u = x^2 + 4$. It follows that $du = 2x\, dx$. The integral is equal to

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln(x^2+4) + C.}$$

Check. The derivative of the proposed answer is

$$\frac{1}{2}\frac{2x}{x^2+4}. \checkmark$$

7. **Find** $\int \frac{1}{x^2+4}\, dx$ **. Check your answer.**

$\boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$.

Check. The derivative of the proposed answer is

$$\frac{1}{2}\frac{\frac{1}{2}}{\left(\frac{x}{2}\right)^2 + 1} = \frac{1}{x^2+4}. \checkmark$$

8. **Find** $\int \frac{1}{\sqrt{x^2+4}}\, dx$ **. Check your answer.**

Let $x = 2\tan\theta$. It follows that $\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4\sec^2} = 2\sec\theta$. It also follows that $dx = 2\sec^2\theta\, d\theta$. Thus, the integral is

$$\int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + C = \ln|\sqrt{x^2+4}+x| - \ln 2 + C = \boxed{\ln|\sqrt{x^2+4}+x| + K},$$

where K is the new constant $-\ln 2 + C$.

Check. The derivative of $\ln(\sqrt{x^2+4}+x)$ is

$$\frac{\frac{2x}{2\sqrt{x^2+4}} + 1}{\sqrt{x^2+4}+x} = \frac{x + \sqrt{x^2+4}}{\sqrt{x^2+4}(\sqrt{x^2+4}+x)} = \frac{1}{\sqrt{x^2+4}}. \checkmark$$

9. **Let** $f(x) = x \ln x$ **. What is the domain of** $f(x)$ **? Where is** $f(x)$ **increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of** $y = f(x)$ **. Graph** $y = f(x)$ **.**

$\boxed{\text{The domain of } f(x) \text{ is all } x > 0.}$ We see that $\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$. The top and bottom both go to infinity, so l'hopital's rule tells us that this limit is $\lim_{x \to 0^+} \frac{1/x}{-1/x^2} \lim_{x \to 0^+} -x = 0$. Thus, $\boxed{\lim_{x \to 0^+} f(x) = 0.}$

We see that $f'(x) = x\frac{1}{x} + \ln x = 1 + \ln x$. Observe that $f'(x)$ is positive for $x < 1/e$; and $f'(x)$ is negative for $0 < x < 1/e$. Thus,

$$\boxed{\begin{array}{c} f(x) \text{ is increasing for } 1/e < x, \\ f(x) \text{ is decreasing for } 0 < x < 1/e, \\ \text{and } (1/e, -1/e) \text{ is a local minimum point on the graph of } y = f(x). \end{array}}$$

We see that $f''(x) = 1/x$, which is always positive. Thus,

$$\boxed{f(x) \text{ is always concave up, never concave down and has no points of inflection.}}$$

The graph appears on another page.

10. **Find** $\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2} - \frac{x^6}{6} - \frac{x^8}{24}}{x^{10}}$ **. Justify your answer.**

We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Thus,

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

and

$$\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2} - \frac{x^6}{6} - \frac{x^8}{24}}{x^{10}} = \lim_{x \to 0} \frac{\frac{x^{10}}{5!} + \frac{x^{12}}{6!} + \dots}{x^{10}} = \lim_{x \to 0} \frac{x^{10}\left[\frac{1}{5!} + \frac{x^2}{6!} + \dots\right]}{x^{10}}$$

$$= \lim_{x \to 0} \frac{1}{5!} + \frac{x^2}{6!} + \dots = \boxed{\frac{1}{5!}}$$

11. **Find** $\int \frac{x\,dx}{(x-3)^2}$ **. Check your answer.**

We multiply both sides of

$$\frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

by $(x-3)^2$ to get

$$x = A(x-3) + B$$

or

$$x = Ax + B - 3A.$$

Equate the corresponding coefficients to see that $A = 1$ and $B = 3$. Check that

$$\frac{1}{x-3} + \frac{3}{(x-3)^2} = \frac{x-3+3}{(x-3)^2} = \frac{x}{(x-3)^2} \checkmark$$

So, the original integral is the same as

$$\int \frac{1}{x-3} + \frac{3}{(x-3)^2}\,dx = \boxed{\ln|x-3| - \frac{3}{x-3} + C.}$$

12. **Find** $\int \ln x\,dx$ **. Check your answer.**

Let $u = \ln x$ and $dv = dx$. It follows that $du = \frac{dx}{x}$ and $v = x$. Integration by parts gives

$$\int u\,dv = uv - \int v\,du = x\ln x - \int x\frac{dx}{x} = \boxed{x\ln x - x + C.}$$

Check. The derivative of the proposed answer is

$$x\frac{1}{x} + \ln x - 1 = \ln x. \checkmark$$

13. **Find** $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$ **.**

The integral is

$$\lim_{b\to\infty} \left.\frac{-1}{\ln x}\right|_e^b = \lim_{b\to\infty} \frac{-1}{\ln b} - \frac{-1}{\ln e} = \frac{1}{\ln e} = \boxed{1}$$

14. **Find the limit of the sequence whose** n^{th} **term is** $a_n = \left(\frac{n-1}{n}\right)^n$ **. Justify your answer.**

We know that $\lim_{n\to\infty}(1+\frac{r}{n})^n = e^r$. It follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{-1}{n}\right)^n = \boxed{e^{-1}.}$$

15. **What familiar function is equal to**

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots ?$$

Justify your answer.

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad \text{for } -1 < x < 1\,.$$

Take the derivative of both sides to see that

$$\boxed{\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \quad \text{for } -1 < x < 1.}$$

16. **Does the series** $\sum_{n=1}^{\infty} \frac{n+3}{n^2\sqrt{n}}$ **converge or diverge? Justify your answer.**

Compare the given series to the converging p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. (The new series has $p = 3/2 > 1$.) Use the limit comparison test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+3}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n+3}{n^2\sqrt{n}} n^{3/2} = \lim_{n\to\infty} \frac{n+3}{n} = \lim_{n\to\infty} 1 + \frac{3}{n} = 1.$$

We see that 1 is a number which is not zero or infinity. It follows that both series converge or both series diverge. We know that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. We conclude that $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges.}}$

17. **Does the series** $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ **converge or diverge? Justify your answer.**

Use the ratio test. Let

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!}\frac{n!}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)}\frac{1}{n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{n^2} = \lim_{n\to\infty} \frac{1}{n} + \frac{1}{n^2} = 0.$$

Thus $\rho < 1$ and $\boxed{\sum_{n=1}^{\infty} \frac{n^2}{n!} \text{ converges.}}$

18. **Find the Taylor polynomial $P_3(x)$ for $f(x) = (1+x)^{3/2}$ about $a = 0$ and bound the error $R_3(x)$ if $-.1 \le x \le 0$.**

We see that

$$\begin{array}{ll} f(x) = (1+x)^{3/2} & f(0) = 1 \\ f'(x) = (3/2)(1+x)^{1/2} & f'(0) = 3/2 \\ f''(x) = (3/4)(1+x)^{-1/2} & f''(0) = 3/4 \\ f'''(x) = (-3/8)(1+x)^{-3/2} & f'''(0) = -3/8 \\ f^{(4)}(x) = (9/16)(1+x)^{-5/2} & \end{array}$$

We know that

$$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!}.$$

So,

$$\boxed{P_3(x) = 1 + (3/2)x + (3/4)\frac{x^2}{2} + (-3/8)\frac{x^3}{3!}.}$$

The remainder

$$|R_3(x)| = \left|\frac{f^{(4)}(c)x^4}{4!}\right| = \left|\frac{9x^4}{16(1+c)^{5/2}4!}\right|$$

for some c with $-.1 \le c \le 0$. We know that $.9 \le 1+c$; so, $\frac{1}{1+c} \le \frac{1}{.9}$. We also know that $|x| < .1$. Thus,

$$\boxed{|R_3(x)| \le \left|\frac{9(.1)^4}{16(.9)^{5/2}4!}\right|.}$$

19. **Use the Parabolic Rule to approximate the amount of water required to fill a pool shaped like the picture below to a depth of 6 feet. (See a different page.) All dimensions are in feet. Recall that Parabolic Rule says that if n is even, then $\int_a^b f(x)dx$ is equal to**

$$\frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right] + E_n,$$

for $h = \frac{b-a}{n}$, $x_i = a + hi$, and $E_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(c)$ for some c with $a \le c \le b$. (Just record the sum. You are not required to perform any addition or multiplication.)

The Parabolic rule gives that the volume is approximately equal to

$$\boxed{6 \cdot \frac{3}{3}\left[22 + 4 \cdot 23 + 2 \cdot 24 + 4 \cdot 23 + 2 \cdot 18 + 4 \cdot 12 + 2 \cdot 10 + 4 \cdot 6 + 0\right] \text{ cubic feet.}}$$

20. **Carbon 14, an isotope of carbon, is radioactive and decays at a rate proportional to the amount present. Its half life is 5730 years; that is, it takes 5730 years for a given amount of carbon 14 to decay to one-half its original size. If 10 grams was present originally, how much will be left after 2000 years? (You may leave** ln **in your answer.)**

Let $A(t)$ be the amount of carbon 14 present at time t, where t is measured in years. Notice that $A(0) = 10$ grams and $A(5730) = 5$ grams. We are supposed to find $A(2000)$. The fact that carbon 14 decays at a rate proportional to the amount present tells us that $A(t) = A(0)e^{kt}$ for some k. Plug in $t = 5730$ to learn that $5 = 10e^{k5730}$. Divide by 10 and take $\ln$ of both sides to see that $\ln(.5) = k5730$ or $\frac{\ln(.5)}{5730} = k$. Our answer is $\boxed{A(2000) = 10e^{\frac{2000\ln(.5)}{5730}} \text{ grams.}}$