1. Find \( \frac{d}{dx} \left( xe^{x^2} \right) \).

\[ x(2xe^{x^2}) + e^{x^2}. \]

2. Find \( \int xe^{x^2} \, dx \). Check your answer.

Let \( u = x^2 \). Then \( du = 2x \, dx \); so the integral is \( \int \frac{1}{2}e^u \, du = \frac{1}{2}e^u + C = \frac{e^{x^2}}{2} + C \).

3. Simplify \( \sin(2 \arccos(\frac{3}{4})) \).

The expression is

\[ 2 \sin(\arccos(\frac{3}{4})) \cos(\arccos(\frac{3}{4})) = \frac{2\sqrt{7}}{4}. \]

Draw a right triangle with the adjacent equal to 3, hypotenuse equal to 4, and opposite equal to \( \sqrt{7} \) to see that \( \sin(\arccos(\frac{3}{4})) = \frac{\sqrt{7}}{4} \).

4. Find \( \int \sin^4 x \cos^3 x \, dx \). Check your answer.

Save one \( \cos x \) turn the other two \( \cos x \)'s into \( \sin x \)'s. Then let \( u = \sin x \). It follows that \( du = \cos x \). The integral is

\[
\int \sin^4 x(1 - \sin^2 x) \cos x \, dx = \int u^4(1 - u^2) \, du = \int (u^4 - u^6) \, du = \frac{u^5}{5} - \frac{u^7}{7} + C
\]

\[ = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C. \]

Check. The derivative of the proposed answer is

\[ \sin^4 x \cos x - \sin^6 x \cos x = \sin^4 x \cos x(1 - \sin^2 x) = \sin^4 x \cos x \cos^2 x. \]
5. **Find** \( \int \sin^2 x \, dx \).

The integral is
\[
\frac{1}{2} \int 1 - \cos 2x \, dx = \frac{1}{2} \left( x - \frac{\sin 2x}{2} \right) + C.
\]

6. **Find** \( \int \frac{x}{x^2 + 4} \, dx \). **Check your answer.**

Let \( u = x^2 + 4 \). It follows that \( du = 2x \, dx \). The integral is equal to
\[
\frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln(x^2 + 4) + C.
\]

**Check.** The derivative of the proposed answer is
\[
\frac{1}{2} \cdot \frac{2x}{2x^2 + 4} = \frac{1}{x^2 + 4}.
\]

7. **Find** \( \int \frac{1}{x^2 + 4} \, dx \). **Check your answer.**

\[
\frac{1}{2} \arctan \left( \frac{x}{2} \right) + C.
\]

**Check.** The derivative of the proposed answer is
\[
\frac{1}{2} \cdot \frac{1}{\left( \frac{x}{2} \right)^2 + 1} = \frac{1}{x^2 + 4}.
\]

8. **Find** \( \int \frac{1}{\sqrt{x^2 + 4}} \, dx \). **Check your answer.**

Let \( x = 2 \tan \theta \). It follows that \( \sqrt{x^2 + 4} = \sqrt{4 \tan^2 \theta + 4} = \sqrt{4 \sec^2 \theta} = 2 \sec \theta \). It also follows that \( dx = 2 \sec^2 \theta \, d\theta \). Thus, the integral is
\[
\int \frac{2 \sec^2 \theta \, d\theta}{2 \sec \theta} = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{\sqrt{x^2 + 4} + x}{2} \right| + C
\]
\[
= \ln \left| \frac{\sqrt{x^2 + 4} + x}{2} \right| + C = \ln |\sqrt{x^2 + 4} + x| - \ln 2 + C = \ln |\sqrt{x^2 + 4} + x| + K,
\]

where \( K \) is the new constant \( -\ln 2 + C \).

**Check.** The derivative of \( \ln(\sqrt{x^2 + 4} + x) \) is
\[
\frac{2x}{\sqrt{x^2 + 4} + x} + 1 = \frac{x + \sqrt{x^2 + 4}}{\sqrt{x^2 + 4}(\sqrt{x^2 + 4} + x)} = \frac{1}{\sqrt{x^2 + 4}},
\]

\( \square \)
9. Let \( f(x) = x \ln x \). What is the domain of \( f(x) \)? Where is \( f(x) \) increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of \( y = f(x) \). Graph \( y = f(x) \).

The domain of \( f(x) \) is all \( x > 0 \). We see that \( \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} \). The top and bottom both go to infinity, so l’hopital’s rule tells us that this limit is \( \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \lim_{x \to 0^+} -x = 0 \). Thus, \( \lim_{x \to 0^+} f(x) = 0 \).

We see that \( f'(x) = x^{1/2} + \ln x = 1 + \ln x \). Observe that \( f'(x) \) is positive for \( x < 1/e \); and \( f'(x) \) is negative for \( 0 < x < 1/e \). Thus,

\[
\begin{align*}
\text{(1/e, -1/e) is a local minimum point on the graph of } y &= f(x). \\
\end{align*}
\]

We see that \( f''(x) = 1/x \), which is always positive. Thus,

\[
\begin{align*}
f(x) \text{ is always concave up, never concave down and has no points of inflection.}
\end{align*}
\]

The graph appears on another page.

10. Find \( \lim_{x \to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2} - \frac{x^6}{6} - \frac{x^8}{24}}{x^{10}} \). Justify your answer.

We know that \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). Thus,

\[
\begin{align*}
e^{x^2} &= \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \\
\end{align*}
\]

and

\[
\begin{align*}
\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2} - \frac{x^6}{6} - \frac{x^8}{24}}{x^{10}} &= \lim_{x \to 0} \frac{x^{10} + \frac{x^{12}}{6!} + \ldots}{x^{10}} = \lim_{x \to 0} \frac{1}{5!} + \frac{x^2}{6!} + \ldots \\
&= \frac{1}{5!}
\end{align*}
\]

11. Find \( \int \frac{x}{(x - 3)^2} \). Check your answer.

We multiply both sides of

\[
x = \frac{A}{x - 3} + \frac{B}{(x - 3)^2}
\]
by \((x - 3)^2\) to get
\[ x = A(x - 3) + B \]
or
\[ x = Ax + B - 3A. \]
Equate the corresponding coefficients to see that \(A = 1\) and \(B = 3\). Check that
\[
\frac{1}{x - 3} + \frac{3}{(x - 3)^2} = \frac{x - 3 + 3}{(x - 3)^2} = \frac{x}{(x - 3)^2} \checkmark
\]
So, the original integral is the same as
\[
\int \frac{1}{x - 3} + \frac{3}{(x - 3)^2} \, dx = \ln |x - 3| - \frac{3}{x - 3} + C.
\]

12. Find \(\int \ln x \, dx\). Check your answer.

Let \(u = \ln x\) and \(dv = dx\). It follows that \(du = \frac{dx}{x}\) and \(v = x\). Integration by parts gives
\[
\int u \, dv = uv - \int v \, du = x \ln x - \int x \frac{dx}{x} = x \ln x - x + C.
\]

Check. The derivative of the proposed answer is
\[
x \frac{1}{x} + \ln x - 1 = \ln x. \checkmark
\]

13. Find \(\int_{e}^{\infty} \frac{1}{x(\ln x)^2} \, dx\).

The integral is
\[
\lim_{b \to \infty} \left[ \frac{-1}{\ln x} \right]_{e}^{b} = \lim_{b \to \infty} \frac{-1}{\ln b} - \frac{-1}{\ln e} = \frac{1}{\ln e} = 1
\]

14. Find the limit of the sequence whose \(n^{th}\) term is \(a_n = \left(\frac{n - 1}{n}\right)^n\). Justify your answer.

We know that \(\lim_{n \to \infty} (1 + \frac{2}{n})^n = e^r\). It follows that
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(\frac{n - 1}{n}\right)^n = \lim_{n \to \infty} \left(1 + \frac{-1}{n}\right)^n = e^{-1}.
\]
15. What familiar function is equal to

\[ f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots ? \]

**Justify your answer.**

We know that

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \text{ for } -1 < x < 1. \]

Take the derivative of both sides to see that

\[ \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \text{ for } -1 < x < 1. \]

16. Does the series \( \sum_{n=1}^{\infty} \frac{n+3}{n^2 \sqrt{n}} \) converge or diverge? **Justify your answer.**

Compare the given series to the converging \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \). (The new series has \( p = 3/2 > 1 \).) Use the limit comparison test.

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n + 3}{n^{3/2}} \cdot \frac{n^2 \sqrt{n}}{n^3} = \lim_{n \to \infty} \frac{n + 3}{n^{3/2}} = \lim_{n \to \infty} \frac{n + 3}{n} = \lim_{n \to \infty} 1 + \frac{3}{n} = 1.
\]

We see that 1 is a number which is not zero or infinity. It follows that both series converge or both series diverge. We know that \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) converges. We conclude that \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) converges.

17. Does the series \( \sum_{n=1}^{\infty} \frac{n^2}{n!} \) converge or diverge? **Justify your answer.**

Use the ratio test. Let

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} \cdot \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} \cdot \frac{n!}{(n+1) \cdot \frac{1}{n^2}} = \lim_{n \to \infty} \frac{(n+1)^2}{(n+1) \cdot \frac{1}{n^2}} = \lim_{n \to \infty} \frac{(n+1) \cdot \frac{1}{n}}{n^2} = \lim_{n \to \infty} \frac{1}{n^2} = 0.
\]

Thus \( \rho < 1 \) and \( \sum_{n=1}^{\infty} \frac{n^2}{n!} \) converges.
18. Find the Taylor polynomial $P_3(x)$ for $f(x) = (1 + x)^{3/2}$ about $a = 0$ and bound the error $R_3(x)$ if $-1 \leq x \leq 0$.

We see that:

$$
\begin{align*}
  f(x) &= (1 + x)^{3/2}, \\
  f'(x) &= (3/2)(1 + x)^{1/2}, \\
  f''(x) &= (3/4)(1 + x)^{-1/2}, \\
  f'''(x) &= (-3/8)(1 + x)^{-3/2}, \\
  f^{(4)}(x) &= (9/16)(1 + x)^{-5/2}.
\end{align*}
$$

We know that:

$$
\begin{align*}
  f(0) &= 1, \\
  f'(0) &= 3/2, \\
  f''(0) &= 3/4, \\
  f'''(0) &= -3/8, \\
  f^{(4)}(0) &= (9/16),
\end{align*}
$$

So,

$$
P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!}.
$$

The remainder

$$
|R_3(x)| = \left|\frac{f^{(4)}(c)x^4}{4!}\right| = \frac{9x^4}{16(1 + c)^{5/2}4!}
$$

for some $c$ with $-1 \leq c \leq 0$. We know that $.9 \leq 1 + c$; so, $\frac{1}{1+c} \leq \frac{1}{3}$. We also know that $|x| < 1$. Thus,

$$
|R_3(x)| \leq \frac{9(1.1)^4}{16(9)^{5/2}4!}.
$$

19. Use the Parabolic Rule to approximate the amount of water required to fill a pool shaped like the picture below to a depth of 6 feet. (See a different page.) All dimensions are in feet. Recall that Parabolic Rule says that if $n$ is even, then $\int_a^b f(x)dx$ is equal to

$$
\frac{h}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right] + E_n,
$$

for $h = \frac{b-a}{n}$, $x_i = a + hi$, and $E_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(c)$ for some $c$ with $a \leq c \leq b$. (Just record the sum. You are not required to perform any addition or multiplication.)

The Parabolic rule gives that the volume is approximately equal to

$$
6 \cdot \frac{3}{3} \left[22 + 4 \cdot 23 + 2 \cdot 24 + 4 \cdot 23 + 2 \cdot 18 + 4 \cdot 12 + 2 \cdot 10 + 4 \cdot 6 + 0\right] \text{ cubic feet.}
$$
20. Carbon 14, an isotope of carbon, is radioactive and decays at a rate proportional to the amount present. Its half life is 5730 years; that is, it takes 5730 years for a given amount of carbon 14 to decay to one-half its original size. If 10 grams was present originally, how much will be left after 2000 years? (You may leave ln in your answer.)

Let $A(t)$ be the amount of carbon 14 present at time $t$, where $t$ is measured in years. Notice that $A(0) = 10$ grams and $A(5730) = 5$ grams. We are supposed to find $A(2000)$. The fact that carbon 14 decays at a rate proportional to the amount present tells us that $A(t) = A(0)e^{kt}$ for some $k$. Plug in $t = 5730$ to learn that $5 = 10e^{k5730}$. Divide by 10 and take ln of both sides to see that $\ln(0.5) = k5730$ or $\frac{\ln(0.5)}{5730} = k$. Our answer is $A(2000) = 10e^{\frac{2000\ln(0.5)}{5730}}$ grams.