Math 142, Exam 2, Fall 2006

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 100 points. There are 10 problems. Each problem is worth 10 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as the exam is graded. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a,b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum\limits_{i=1}^{n} f(x_i^*)\Delta_i$.

2. **Find** $\int xe^x\,dx$. **Check your answer.**

Use integration by parts with $u = x$ and $dv = e^x dx$. It follows that $du = dx$ and $v = e^x$. The integration by parts formula is

$$\int udv = uv - \int vdu.$$

Our integral is equal to

$$xe^x - \int e^x dx = \boxed{xe^x - e^x + C.}$$

Check. The derivative of the proposed answer is

$$xe^x + e^x - e^x = xe^x. \checkmark$$

3. **Find** $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$. **Check your answer.**

We do a Trig substitution. Let $x = \tan\theta$. It follows that $dx = \sec^2\theta\, d\theta$,

$$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = \sec\theta,$$

and the integral is

$$\int \frac{\sec^2\theta}{\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \boxed{\ln|\sqrt{x^2+1} + x| + C.}$$

Check: The derivative of the proposed answer is

$$\frac{\frac{2x}{2\sqrt{x^2+1}} + 1}{\sqrt{x^2+1}+x} = \frac{\frac{x}{\sqrt{x^2+1}} + 1}{\sqrt{x^2+1}+x} = \frac{\left(\frac{x}{\sqrt{x^2+1}} + 1\right)\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)\sqrt{x^2+1}} = \frac{x + \sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)\sqrt{x^2+1}}$$

$$= \frac{1}{\sqrt{x^2+1}}. \checkmark$$

4. **Find** $\displaystyle\int \frac{xdx}{\sqrt{1+x^2}}$. **Check your answer.**

Let $u = 1+x^2$. It follows that $du = 2xdx$. The integral is $\frac{1}{2}\int u^{-\frac{1}{2}}du = \sqrt{u}+C = \sqrt{1+x^2}+C$.

Check. The derivative of the proposed answer is $\frac{1}{2}\frac{1}{\sqrt{1+x^2}}2x\checkmark$

5. **Find** $\int \sin^5 x\, dx$. **Check your answer.**

Save one $\sin x$ convert the other $\sin x$'s into $\cos x$. The integral is

$$\int (1-\cos^2 x)^2 \sin x\, dx.$$

Let $u = \cos x$, so $du = -\sin x$. The integral is

$$-\int (1-u^2)^2\, du = -\int (1-2u^2+u^4)\, du = -\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + C =$$

$$= \boxed{-\left(\cos x - \frac{2\cos^3 x}{3} + \frac{\cos^5 x}{5}\right) + C}$$

Check. The derivative of the proposed answer is

$$-(-\sin x - 2\cos^2 x(-\sin x) + \cos^4 x(-\sin x)) = -(-\sin x)(1 - 2\cos^2 x + \cos^4 x)$$

$$= \sin x(1-\cos^2 x)^2 \checkmark.$$

6. **Find** $\int \sin^2 2x\, dx$.

Use the identity $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ to see that the integral is equal to $\frac{1}{2}\int (1-\cos 4x)\, dx = \boxed{\frac{1}{2}\left(x - \frac{\sin 4x}{4}\right) + C}$

7. **Find the area between $x = y^2$ and $8 = x + 2y$. (There is no need for you to do arithmetic. Leave your answer in terms of sums and products of numbers.)**

I drew a picture else where. I partition the y-axis from $y = -4$ to $y = 2$. I approximate the area using boxes of base dy and length $(8-2y) - y^2$. The area is

$$\int_{-4}^{2} 8 - 2y - y^2\, dy = \left(8y - y^2 - \frac{y^3}{3}\right)\Bigg|_{-4}^{2}$$

$$= \boxed{8(2) - 4 - \frac{8}{3} - \left(8(-4) - (16) - \frac{(-4)^3}{3}\right)}.$$

8. **Find the length of the curve**

$$\begin{cases} x = \frac{1}{3}t^3 \\ y = \frac{1}{2}t^2, \end{cases}$$

for $0 \le t \le 1$ **.**

The arc length is equal to

$$\int_{\text{beginning}}^{\text{end}} ds = \int_{\text{beginning}}^{\text{end}} \sqrt{dx^2 + dy^2} = \int_{\text{beginning}}^{\text{end}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{(t^2)^2 + (t)^2}\, dt = \int_0^1 \sqrt{t^4 + t^2}\, dt = \int_0^1 t\sqrt{t^2+1}\, dt = \tfrac{1}{2}\tfrac{2}{3}(t^2+1)^{\frac{3}{2}}\Big|_0^1$$

$$= \tfrac{1}{2}\tfrac{2}{3}(2^{\frac{3}{2}} - 1) = \boxed{\tfrac{1}{3}(2^{\frac{3}{2}} - 1)}.$$

9. **Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the x-axis are equilateral triangles. There is no revolution in this problem.**

We find the volume of the solid by using the slicing method. I drew a picture elsewhere. Chop the x-axis from $x = 0$ to $x = 1$ into small pieces. Fix the small piece of the x-axis with x-coordinate x. Consider the little line segment above our piece of the x-axis whose bottom has y-ccodinate x^2 and whose top has y-coordinate x. This segment is the base of one slice of our solid. The slice in question is an equilateral triangle. The volume of the slice is the area of the triangle times the tickness, which is $\frac{1}{2}bht$, where $b = x - x^2$ and $t = dx$. Draw an equilateral triangle to see that $h = \frac{\sqrt{3}}{2}b$. The volume of the slice is $\frac{1}{2}bht = \frac{\sqrt{3}}{4}(x-x^2)^2 dx$. The volume of the solid is

$$\frac{\sqrt{3}}{4}\int_0^1 (x-x^2)^2 dx = \frac{\sqrt{3}}{4}\int_0^1 (x^2 - 2x^3 + x^4) dx = \frac{\sqrt{3}}{4}\left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5})\right)\Big|_0^1$$

$$= \frac{\sqrt{3}}{4}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \boxed{\frac{\sqrt{3}}{120}}.$$

10. **The vat shown in the accompanying figure contains water to a depth of 2 m. Find the work required to pump all the water to the top of the vat. [Use $9810\ N/m^3$ as the weight density of water.] (There is no need for you to do arithmetic. Leave your answer in terms of sums and products of numbers.) Be sure to give the correct units.**

I draw my axis with $y = 0$ at the top of the tank. At the beginning the water sits between $y = 1$ and $y = 3$. I will move this water to $y = 0$. Chop the y-axis from $y = 1$ to $y = 3$ into small pieces. The work required to move the layer of water with y-coordinate y is

(the weight of the layer)(the distance the layer is moved).

The layer looks like a rectangular slab with volume ℓwt, where $\ell = 6$, w depends on y and $t = dy$. Similar triangles shows that $w = \frac{4}{3}(3 - y)$. (This makes sense because when $y = 0$, then $w = 4$; and when $y = 3$, $w = 0$.) The work required to move the layer of water with y-coordinate y is

(the weight of the layer)(the distance the layer is moved)

= (the volume of the layer)(the density of water)(the distance the layer is moved)

$$= \ell wt(9810)y = 6 \cdot \tfrac{4}{3}(3 - y) \cdot dy \cdot 9810y.$$

The total work to empty the tank is

$$6(9810)\tfrac{4}{3}\int_1^3 (3 - y)y\,dy = 6(9810)\tfrac{4}{3}\int_1^3 (3y - y^2)dy = 6(9810)\tfrac{4}{3}\ \tfrac{3y^2}{2} - \tfrac{y^3}{3}\Big|_1^3$$

$$= \boxed{6(9810)\tfrac{4}{3}(\tfrac{27}{2} - 9 - \tfrac{3}{2} + \tfrac{1}{3})\text{ Joules.}}$$