Math 142, Exam 3, Solutions, Fall 2015

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.
The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer.
No Calculators or Cell phones.

1. **Find** $\displaystyle\int \frac{-x^2+6x-3}{(x-1)^3}dx$ **. Please check your answer.**

We use the method of partial fractions and look for integers A, B, and C with

$$\frac{-x^2+6x-3}{(x-1)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}.$$

Multiply both sides by $(x-1)^3$ to obtain

$$-x^2+6x-3=A(x-1)^2+B(x-1)+C$$

$$-x^2+6x-3=Ax^2-2Ax+A+Bx-B+C$$

$$-x^2+6x-3=Ax^2+(-2A+B)x+(A-B+C).$$

Equate the corresponding coefficients to see $-1=A$, $6=-2A+B$, and $-3=A-B+C$. Thus, $A=-1$, $6=2+B$ (so, $B=4$) and $-3=-1-4+C$ (so, $2=C$). At this point we seem to have shown that

$$\frac{-x^2+6x-3}{(x-1)^3}=\frac{-1}{x-1}+\frac{4}{(x-1)^2}+\frac{2}{(x-1)^3}.$$

We make sure this is correct before continuing. The right side is

$$\frac{-1}{x-1}+\frac{4}{(x-1)^2}+\frac{2}{(x-1)^3}=\frac{-(x^2-2x+1)+4(x-1)+2}{(x-1)^3}=\frac{-x^2+6x-3}{(x-1)^3},$$

which is the left side. Now the integral is easy:

$$\int \frac{-x^2+6x-3}{(x-1)^3}dx=\int\left(\frac{-1}{x-1}+\frac{4}{(x-1)^2}+\frac{2}{(x-1)^3}\right)dx$$

$$=\boxed{-\ln|x-1|-\frac{4}{x-1}-\frac{1}{(x-1)^2}+C.}$$

2. **Find** $\displaystyle\int_1^7 \frac{dx}{(x-3)^2}$ **. Please draw a meaningful picture.**

Please look at the picture on the other page. The picture shows us that the answer is either some positive number or $+\infty$. We compute

$$\int_1^7 \frac{dx}{(x-3)^2} = \lim_{a\to 3^-}\int_1^a \frac{dx}{(x-3)^2} + \lim_{b\to 3^+}\int_b^7 \frac{dx}{(x-3)^2}$$

$$= \lim_{a\to 3^-} \frac{-1}{x-3}\bigg|_1^a + \lim_{b\to 3^+} \frac{-1}{x-3}\bigg|_b^7 = \lim_{a\to 3^-}\left(\frac{-1}{a-3} - \frac{-1}{1-3}\right) + \lim_{b\to 3^+}\left(\frac{-1}{7-3} - \frac{-1}{b-3}\right)$$

$$= +\infty - \frac{1}{2} - \frac{1}{4} + \infty = +\infty.$$

$$\boxed{\text{The integral diverges to } +\infty.}$$

3. **Find the volume of the solid which is obtained by revolving the region bounded by $x = y^2$ and $y + x = 2$ about the line $y = -5$. Please draw a meaningful picture. It is not necessary for you to do the integral.**

Please look at the picture on the other page. The two curves intersect when $y + y^2 = 2$; so $y^2 + y - 2 = 0$, or $(y-1)(y+2) = 0$. The points of intersection are $(1,1)$ and $(4,-2)$. Chop the y-axis from $y = -2$ to $y = 1$. Rotate the rectangle with y-coordinate y . Obtain a shell of volume $2\pi rht$, where $t = dy$, $r = y + 5$, and $h = 2 - y - y^2$. The volume of the shell is

$$2\pi rht = 2\pi(y+5)(2-y-y^2)dy.$$

The volume of the solid is

$$\boxed{2\pi\int_{-2}^{1}(y+5)(2-y-y^2)dy}.$$

4. **What is the limit of the sequence whose n^{th} term is $a_n = (\frac{n-3}{n})^{4n}$. Please explain your answer.**

Maybe you know that $\lim\limits_{n\to\infty}(1+\frac{r}{n})^n = e^r$. If so, you could say:

$$\lim_{n\to\infty} a_n = \left(\lim_{n\to\infty}\left(1+\frac{-3}{n}\right)^n\right)^4 = (e^{-3})^4 = e^{-12}.$$

Otherwise,

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \ln\left(\frac{n-3}{n}\right)^{4n} = \lim_{n\to\infty} 4n(\ln\left(\frac{n-3}{n}\right)) = \lim_{n\to\infty} \frac{4(\ln(1-\frac{3}{n}))}{\frac{1}{n}}.$$

The top and the bottom both go to ∞ ; we use L'hopital's rule.

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\frac{4(\frac{3}{n^2})}{(1-\frac{3}{n})}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{4(-3)}{(1-\frac{3}{n})} = -12.$$

It follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{-12}.$$

$$\boxed{\text{The sequence converges to } e^{-12}.}$$

5. **Please express the repeating decimal $d = 2.1497979797\ldots$ as a ratio of two integers. Please explain your answer.**

We see that $100d - d = 214.979797\cdots - 2.14979797\ldots$; so $99d = 212.83$ and

$$d = \frac{212.83}{99} = \boxed{\frac{21283}{9900}}.$$