Math 142, Exam 3 Solutions, Fall 2011

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **7** problems on **2** sides.

No Calculators or Cell phones. Write in complete sentences. Explain what you are doing VERY thoroughly.

1. (8 points) **Consider the sequence defined by** $a_1 = 2$ **and** $a_{n+1} = \frac{1}{4-a_n}$ **.**
   (a) Prove that $0 < a_n \le 2$ **for all positive integers** n **.**
   (b) Prove that $a_{n+1} \le a_n$ **for all positive integers** n **.**
   (c) State the Completeness Axiom and draw a conclusion about the sequence $\{a_n\}$ **from the Completeness Axiom.**
   (d) Find the limit of the sequence $\{a_n\}$ **.**

(a) We use the technique of Mathematical Induction. We see that $a_1 = 2$ and therefore, $0 < a_1 \le 2$. Assume **BY INDUCTION** that $0 < a_{n-1} \le 2$ for some **FIXED** n. Multiply by -1 to see $-2 \le -a_{n-1} < 0$. Add 4 to see $2 \le 4 - a_{n-1} < 4$; that is $2 \le 4 - a_{n-1}$ and $4 - a_{n-1} < 4$. Divide the first inequality by the positive number $2(4 - a_{n-1})$ to obtain $\frac{1}{4-a_{n-1}} \le \frac{1}{2}$. Divide the second inequality by the positive number $(4 - a_{n-1})4$ to see $\frac{1}{4} < \frac{1}{4-a_{n-1}}$. Put the inequalities back together to see: $\frac{1}{4} < \frac{1}{4-a_{n-1}} \le \frac{1}{2}$. We have shown that

$$0 < a_{n-1} \le 2 \implies \frac{1}{4} < \frac{1}{4-a_{n-1}} \le \frac{1}{2}.$$

Obviously, $\frac{1}{4-a_{n-1}} = a_n$, $0 < \frac{1}{4}$ and $\frac{1}{2} \le 2$; so,

$$0 < a_{n-1} \le 2 \implies 0 < a_n \le 2.$$

We saw that $0 < a_1 \le 2$ for $n = 1$. We proved that if $0 < a_{n-1} \le 2$ for some FIXED n, then $0 < a_n \le 2$ also holds for that one FIXED n. We apply the Principle of Mathematical Induction to conclude that $0 < a_n \le 2$ for ALL positive integers n.

(b) We use the technique of Mathematical Induction. We see that $a_1 = 2$ and $a_2 = \frac{1}{2}$; so $a_2 \le a_1$. Assume **BY INDUCTION** that $a_n \le a_{n-1}$ for some **FIXED** n. Add $-a_n - a_{n-1}$ to both sides to see $-a_{n-1} \le -a_n$. Add 4 to both sides to see: $4 - a_{n-1} \le 4 - a_n$. Both numbers are positive because part (1) shows that $a_n \le 2$ for all n. Divide both sides by the positive number $(4-a_{n-1})(4-a_n)$ to obtain $\frac{1}{4-a_n} \le \frac{1}{4-a_{n-1}}$ and this is $a_{n+1} \le a_n$. Thus

$$a_n \le a_{n-1} \implies a_{n+1} \le a_n.$$

We saw that $a_{n+1} \le a_n$ for $n = 1$. We proved that if $a_n \le a_{n-1}$ for some FIXED n, then $a_{n+1} \le a_n$ also holds for that one FIXED n. We apply the Principle of Mathematical Induction to conclude that $a_{n+1} \le a_n$ for ALL positive integers n.

(c) The completeness axiom says that every decreasing bounded sequence of real numbers has a limit. We showed in (1) and (2) that $\{a_n\}$ is an decreasing bounded sequence of real numbers. We conclude that $\lim\limits_{n\to\infty} a_n$ exists. Let $L = \lim\limits_{n\to\infty} a_n$.

(d) Take $\lim\limits_{n\to\infty}$ of both sides of $a_{n+1} = \frac{1}{4-a_n}$ to conclude that

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{4 - \lim\limits_{n\to\infty} a_n};$$

that is, $L = \frac{1}{4-L}$; so $L(4-L) = 1$ or $-L^2 + 4L = 1$. We use the quadratic formula to solve $0 = L^2 - 4L + 1$. We obtain $L = \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$. We know that L can not be more than 2 because every term in the sequence is less than or equal to 2. So $L \ne 2+\sqrt{3}$ and hence L does equal $2-\sqrt{3}$.

2. (7 points) **Find the limit of the sequence whose n^{th} term is $a_n = (\frac{n-3}{n})^{2n}$.**

We learned in class that $\lim\limits_{n\to\infty}(1+\frac{r}{n})^n = e^r$. Thus,

$$\lim_{n\to\infty} = \lim_{n\to\infty}\left(\frac{n-3}{n}\right)^{2n} = \left(\lim_{n\to\infty}\left(1+\frac{-3}{n}\right)^{n}\right)^2 = \left(e^{-3}\right)^2 = \boxed{e^{-6}}.$$

3. (7 points) **Consider the series $\sum\limits_{k=3}^{\infty} 6(\frac{1}{3})^k$. Does the series converge? Find the sum of the series if possible. Explain what you are doing in great detail.**

This series is the geometric series with initial term $a = 6(\frac{1}{3})^3$ and ratio $r = \frac{1}{3}$. We know that if $|r| < 1$, then the geometric series with initial term a and ratio r converges to

$$\frac{a}{1-r} = \boxed{\frac{6(\frac{1}{3})^3}{1-\frac{1}{3}}}.$$

4. (7 points) **Consider the series** $\sum_{k=2}^{\infty}(\frac{1}{k} - \frac{1}{k+2})$**. For each integer** n**, with** $2 \le n$**, let** $s_n = \sum_{k=2}^{n}(\frac{1}{k} - \frac{1}{k+2})$

(a) Write down s_5**. Be sure to cancel everything that cancels.**

(b) Find a closed formula for s_n**. Recall that a closed formula does not have any summation signs or any dots.**

(c) Find $\lim_{n\to\infty} s_n$**.**

(d) Does the series $\sum_{k=2}^{\infty}(\frac{1}{k} - \frac{1}{k+2})$ **converge?**

(e) Find the sum of the series $\sum_{k=2}^{\infty}(\frac{1}{k} - \frac{1}{k+2})$**, if possible.**

(a)

$$s_5 = \sum_{k=2}^{5}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right)$$

$$= \boxed{\frac{1}{2} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7}}.$$

(b)

$$s_n = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \boxed{\frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2}}.$$

(c)

$$\lim_{n} s_n = \lim_{n} \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} = \boxed{\frac{1}{2} + \frac{1}{3}}$$

(d) and (e) Yes the series $\sum_{k=2}^{\infty}(\frac{1}{k} - \frac{1}{k+2})$ converges and the sum is $\boxed{\frac{1}{2} + \frac{1}{3}}$.

5. (7 points) **Estimate** $\sum_{k=1}^{\infty} \frac{1}{k^5}$ **with an error at most** $\frac{1}{1000}$**.**

We estimate $\sum_{k=1}^{\infty} \frac{1}{k^5}$ by $\sum_{k=1}^{N} \frac{1}{k^5}$ for some integer N which we now determine. The distance between $\sum_{k=1}^{\infty} \frac{1}{k^5}$ and $\sum_{k=1}^{N} \frac{1}{k^5}$ is

$$\sum_{k=N+1}^{\infty} \frac{1}{k^5}.$$

Look at the picture, to see that

$$\sum_{k=N+1}^{\infty} \frac{1}{k^5} = \text{the area inside the boxes} \leq \text{the area under the curve} = \int_N^{\infty} \frac{1}{x^5} dx$$

$$= \lim_{b\to\infty} \frac{-1}{4x^4}\bigg|_N^b = \lim_{b\to\infty} \left(\frac{-1}{4b^4} + \frac{1}{4N^4}\right) = \frac{1}{4N^4}.$$

We want the distance between $\sum_{k=1}^{\infty} \frac{1}{k^5}$ and $\sum_{k=1}^{N} \frac{1}{k^5}$ to be at most $\frac{1}{1000}$; so we make $\frac{1}{4N^4} \leq \frac{1}{1000}$. We make $\frac{1000}{4} \leq N^4$. We make $250 \leq N^4$. We make $4 \leq N$. We conclude that $\sum_{k=1}^{4} \frac{1}{k^5}$ approximates $\sum_{k=1}^{\infty} \frac{1}{k^5}$ with an error at most $1/1000$.

6. (7 points) **Does the series $\sum_{k=1}^{\infty} \frac{1}{k-\ln k}$ converge? Justify your answer VERY thoroughly.**

We compare the given series to the divergent Harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. We see that $\frac{1}{k} \leq \frac{1}{k-\ln k}$. Part (b) of the comparison test shows that $\sum_{k=1}^{\infty} \frac{1}{k-\ln k}$ also diverges.

7. (7 points) **Does the series $\sum_{k=1}^{\infty} \frac{k}{2k+3}$ converge? Justify your answer VERY thoroughly.**

We see that $\lim_{k\to\infty} \frac{k}{2k+3} = \lim_{k\to\infty} \frac{1}{2+\frac{3}{k}} = \frac{1}{2} \neq 0$. Apply the Individual Term Test For Divergence to conclude that $\sum_{k=1}^{\infty} \frac{k}{2k+3}$ diverges.