# Math 142, Exam 3, Fall 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer.

**No Calculators or Cell phones.**

(1) (9 points) **Find** $\int \frac{1}{\sqrt{x^2+1}}\,dx$. **Please check your answer.**

Let $x = \tan\theta$. Compute that

$$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = \sec\theta,$$

and that $dx = \sec^2\theta d\theta$. The original integral is equal to

$$\int \frac{\sec^2\theta d\theta}{\sec\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \boxed{\ln|\sqrt{x^2+1}+x|+C}.$$

**Check.** We see that

$$\frac{d}{dx}(\ln(\sqrt{x^2+1}+x) = \frac{\frac{2x}{2\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x} = \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}(\sqrt{x^2+1}+x)} = \frac{1}{\sqrt{x^2+1}}.\checkmark$$

(2) (9 points) **Find** $\int_0^4 \frac{1}{(x-3)^2}\,dx$.

The function $f(x) = \frac{1}{(x-3)^2}$ becomes infinite at $3$ which is between $0$ and $4$; so we will integrate from $0$ to a little short of $3$ and from a little more than $3$ until $4$. Once we have completed the above integrals, we will take two limits. We drew a picture on the picture page. The original problem is equal to

$$\lim_{b\to 3^-}\int_0^b \frac{1}{(x-3)^2}\,dx + \lim_{a\to 3^+}\int_a^4 \frac{1}{(x-3)^2}\,dx = \lim_{b\to 3^-} \frac{-1}{(x-3)}\Big|_0^b + \lim_{a\to 3^+} \frac{-1}{(x-3)}\Big|_a^4$$

$$= \lim_{b\to 3^-}\left(\frac{-1}{(b-3)} - \frac{-1}{-3}\right) + \lim_{a\to 3^+}\left(\frac{-1}{(4-3)} - \frac{-1}{(a-3)}\right)$$

$$= +\infty - \frac{1}{3} - 1 + \infty = \boxed{+\infty}$$

(3) (8 points) **Find the limit of the sequence whose $n^{\text{th}}$ term is** $a_n = \left(\frac{2n-1}{2n}\right)^n$.

We know that $\lim_{n\to\infty}(1+\frac{r}{n})^n = e^r$. We compute

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{-1/2}{n}\right)^n = \boxed{e^{-1/2}.}$$

(4) (8 points) **Find a closed formula for the sum**

$$\sum_{k=2}^{100} \pi^k = \pi^2 + \pi^3 + \pi^4 + \cdots + \pi^{99} + \pi^{100}.$$

**Remember that a closed formula does not have any summation signs or any dots. Be sure to give a formula for the given sum and not something else.**

Let $S = \pi^2 + \pi^3 + \pi^4 + \cdots + \pi^{99} + \pi^{100}$. Observe that

$$\pi S = \pi^3 + \pi^4 + \cdots + \pi^{99} + \pi^{100} + \pi^{101}.$$

It follows that

$$(\pi - 1)S = \pi S - S = \pi^{101} - \pi^2.$$

We conclude that

$$S = \frac{\pi^{101} - \pi^2}{(\pi - 1)}.$$

Thus,

$$\boxed{\pi^2 + \pi^3 + \pi^4 + \cdots + \pi^{99} + \pi^{100} = \frac{\pi^{101} - \pi^2}{(\pi - 1)}.}$$

(5) (8 points) **Approximate** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5}$ **with an error at most** $\frac{4}{10^4}$. **Explain what you are doing. Write in complete sentences.**

Look at the picture on the picture page. We approximate the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k^5}$ with the finite sum $\sum_{k=1}^{N} \frac{1}{k^5}$. We will pick $N$ large enough so that the error that is incurred when this approximation is made is at most $\frac{4}{10^4}$. We see that the error is equal to

$$\left| \sum_{k=1}^{\infty} \frac{1}{k^5} - \sum_{k=1}^{N} \frac{1}{k^5} \right| = \sum_{k=N+1}^{\infty} \frac{1}{k^5}$$

$$= \text{the area inside the boxes} < \text{the area under the curve}$$

$$= \int_N^{\infty} \frac{1}{x^5} dx = \lim_{b \to \infty} \frac{-1}{4x^4}\bigg|_N^b = \lim_{b \to \infty} \left( \frac{-1}{4b^4} + \frac{1}{4N^4} \right) = \frac{1}{4N^4}.$$

We want the error to be at most $\frac{4}{10^4}$. We have shown that the error is at most $\frac{1}{4N^4}$; so, we make

$$\frac{1}{4N^4} \le \frac{4}{10^4}.$$

The most recent inequality is equivalent to $\frac{10^4}{16} \le N^4$ and this is equivalent to $\frac{10}{2} \le N$. Of course, $\frac{10}{2} = 5$. We conclude that

$$\boxed{\sum_{k=1}^{5} \frac{1}{k^5} \text{ approximates } \sum_{k=1}^{\infty} \frac{1}{k^5} \text{ with an error at most } \frac{4}{10^4}}.$$

(6) (8 points) **Does the series** $\sum_{k=1}^{\infty} \frac{1}{k^2+5}$ **converge? Justify your answer very thoroughly. Write in complete sentences.**

We use the comparison test. We see that $k^2 \leq k^2 + 5$; hence,

$$\frac{1}{k^2+5} \leq \frac{1}{k^2}$$

and both numbers are positive. The p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges because $p = 2$ and $1 < 2$. Thus,

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2+5} \text{ also converges by part (a) of the comparison test.}}$$