Math 142, Exam 3, Solutions, Spring 2006

There are 10 problems. Each problem is worth 10 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Find** $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$ **. Check your answer.**

The answer is $\boxed{\arcsin e^x + C}$. The derivative of the proposed answer is $\frac{e^x}{\sqrt{1-e^{2x}}}$.

2. **Find** $\int \sec^2 x \tan x dx$ **. Check your answer.**

The answer is $\boxed{\frac{\tan^2 x}{2} + C}$. The derivative of the proposed answer is $\sec^2 x \tan x$

3. **Find** $\int \frac{1}{\sqrt{x^2-9}}dx$ **. Check your answer.**

Let $x = 3\sec\theta$. It follows that $dx = 3\sec\theta\tan\theta d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$. The integral is

$$\int \frac{3\sec\theta\tan\theta d\theta}{3\tan\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$

$$= \boxed{\ln|x + \sqrt{x^2-9}| + K},$$

where $C - \ln 3 = K$. The derivative of the proposed answer is

$$\frac{1 + \frac{x}{\sqrt{x^2-9}}}{x + \sqrt{x^2-9}}.$$

Multiply top and bottom by $\sqrt{x^2-9}$ to see that this is exactly equal to $\frac{1}{\sqrt{x^2-9}}$.

4. **Find** $\int \sec^3 x dx$ **. Check your answer.**

Use integration by parts. Let $u = \sec x$ and $dv = \sec^2 x dx$. We see that $du = \sec x \tan x dx$, $v = \tan x$, and

$$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx.$$

Add $\int \sec^3 x dx$ to both sides to see that

$$2\int \sec^3 x dx = \sec x \tan x + \ln|\sec x + \tan x| + C.$$

We conclude that

$$\boxed{\int \sec^3 x dx = (1/2)[\sec x \tan x + \ln|\sec x + \tan x|] + C.}$$

The derivative of the proposed answer is

$$(1/2)\left[\sec^3 x + \sec x \tan^2 x + \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\right]$$

$$= (1/2)[\sec^3 x + \sec x(\sec^2 x - 1) + \sec x]. \checkmark$$

5. **Find** $\int \frac{3x-2}{2x^2-3x+1} dx$ **. Check your answer.**

Use the technique of partial fractions to see that

$$\frac{3x-2}{2x^2-3x+1} = \frac{1}{x-1} + \frac{1}{2x-1}.$$

The integral is $\boxed{\ln|x-1| + \frac{1}{2}\ln|2x-1| + C}$.

6. **Find** $\int \frac{4x^2+2x+3}{x^3+x} dx$ **. Check your answer.**

Use the technique of partial fractions to see that

$$\frac{4x^2+2x+3}{x^3+x} = \frac{x+2}{x^2+1} + \frac{3}{x}.$$

The integral is

$$\boxed{(1/2)\ln(x^2+1) + 2\arctan x + 3\ln|x| + C.}$$

7. **Find** $\int_{-1}^{4} \frac{1}{x^2} dx$ **.**

The integral is equal to

$$\lim_{b \to 0^-} \int_{-1}^{b} \frac{1}{x^2} dx + \lim_{a \to 0^+} \int_{a}^{4} \frac{1}{x^2} dx$$

$$= \lim_{b \to 0^-} \left. \frac{-1}{x} \right|_{-1}^{b} + \lim_{a \to 0^+} \left. \frac{-1}{x} \right|_{a}^{4} = \lim_{b \to 0^-} \frac{-1}{b} - 1 + \lim_{a \to 0^+} -\frac{1}{4} + \frac{1}{a} = +\infty + \infty = +\infty.$$

Any finite answer, especially a negative finite answer, is worth 0.

8. **Consider the sequence whose** n^{th} **term is** $a_n = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{n^3}$ **. Approximate** a_n **by an integral in a meaningful manner. Be sure to make clear which quantity is larger. Does the sequence** $\{a_n\}$ **converge?**

I drew a picture on another page. The area inside the boxes is $\frac{1}{2^3} + \cdots + \frac{1}{n^3} = a_n - 1$. The area under the curve is $\int_1^n \frac{1}{x^3} dx$. The area inside the boxes is less than the area under the curve. We conclude that $a_n - 1 \le \int_1^n \frac{1}{x^3} dx$. Thus,

$$a_n \le 1 + \int_1^n \frac{1}{x^3} dx = \left. \frac{1}{-2x^2} \right|_1^n + 1 = 1/2 - \frac{1}{2n^2} + 1 \le 3/2.$$

We conclude that $a_n \le 3/2$ for all n . The sequence $\{a_n\}$ is increasing and bounded from above. The completeness axiom guarantees that the sequence converges.

9. **Consider the sequence** $\{a_n\}$ **with** $a_1 = \sqrt{12}$ **, and** $a_n = \sqrt{12 + a_{n-1}}$ **for** $n \ge 2$ **. Prove that the sequence** $\{a_n\}$ **converges. Find the limit of the sequence** $\{a_n\}$ **.**

It is clear that every term a_n is at most 4 . We see that $a_1 \le 4$. If $a_{n-1} \le 4$, then $a_{n-1} + 12 \le 16$; so $a_n = \sqrt{a_{n-1} + 12} < \sqrt{16} = 4$. It is also clear that the sequence is an increasing sequence. We just saw that $a_n \le 4$ for all n . Multiply both sides by the positive number $a_n + 3$ to see that $a_n^2 + 3a_n \le 4a_n + 12$. In other words, we have $a_n^2 \le a_n + 12$. The numbers a_n and $a_n + 12$ are both positive. it follows that $a_n \le \sqrt{a_n + 12} = a_{n+1}$. The sequence $\{a_n\}$ is an increasing bounded sequence. The completeness axiom guarantees that this sequence converges. We know that $\lim\limits_{n \to \infty} a_n$ exists. Let L be the name of this limit. Take the limit of both sides of $a_n = \sqrt{12 + a_{n-1}}$ to see that $L = \sqrt{12 + L}$, or $L^2 = 12 + L$, which is $L^2 - L - 12 = 0$. This equation factors to become $(L - 4)(L + 3) = 0$; hence $L = 4$ or $L = -3$. Every a_n is positive so $L = -3$ is not possible. We conclude that $\boxed{\lim\limits_{n \to \infty} a_n = 4}$.

10. **A conical water tank sits with its base on the ground. The radius of the base is 10 feet. The height of the tank is 30 feet. The tank is filled to a depth of 25 feet. How much work is required to pump all of the water out through a hole in the top of the tank? The density of water is 62.4 lb/ft ^{3}. Be sure to give the units for your answer.**

I drew a picture elsewhere. Notice that I arranged my axis, so that $x = 0$ is the top of the tank. The water starts at $x = 5$. The bottom of the water occurs at $x = 30$. For each x between 5 and 30, we lift a thin layer of water starting at x-coordinate x. The work to lift this thin layer is the weight of the layer times the distance this layer must be lifted. The distance is x. (That is the advantage of the way I set my axis.) The weight of the layer is the volume of the layer times the density of water. The volume of the layer is the area of the top times the thickness. The thickness is dx and the area of the top is πr^2, where similar triangles tell us that $r = \frac{1}{3}x$. The work to lift the layer of water at x-coordinate x is

$$(62.4)\pi(\tfrac{1}{3}x)^2 x dx.$$

The total work is

$$\frac{(62.4)\pi}{9}\int_5^{30} x^3 dx = \frac{(62.4)\pi}{9}\frac{x^4}{4}\bigg|_5^{30} = \boxed{\frac{(62.4)\pi}{36}[30^4 - 5^4] \text{ foot-pounds.}}$$