Math 142, Exam 2, Solutions, Fall 2011

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible.

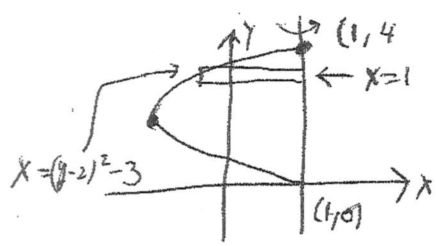
No Calculators or Cell phones.

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a,b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum_{i=1}^{n} f(x_i^*)\Delta_i$.

2. (6 points) **Consider the region bounded by $x+3 = (y-2)^2$ and $x = 1$. Revolve this region about the line $x = 1$. Find the volume of the resulting solid. You must draw a meaningful picture.**

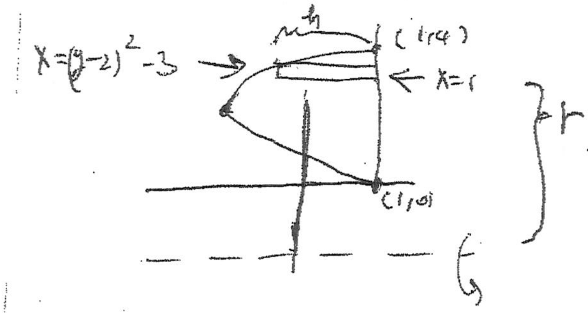
The intersection points occur when $4 = (y-2)^2$; so $2 = y-2$ or $-2 = y-2$.



Chop the y-axis from $y = 0$ to $y = 4$. Consider the rectangle with y-coordinate y. Revolve the rectangle about $x = 1$. Get a disk of volume $\pi r^2 t$, with $t = dy$, and $r = 1 - ((y-2)^2 - 3) = 1 - (y^2 - 4y + 4 - 3) = 4y - y^2$. The vol of the disk is $\pi r^2 t = \pi(4y - y^2)^2 dy$. The volume of the solid is

$$\pi \int_0^4 (16y^2 - 8y^3 + y^4)dy = \pi(16y^3/3 - 2y^4 + y^5/5)\Big|_0^4 = \boxed{\pi(16(64)/3 - 2(4)^4 + 4^5/5)}$$

3. (6 points) **Consider the region bounded by $x+3=(y-2)^2$ and $x=1$. Revolve this region about the line $y=-1$. Find the volume of the resulting solid. You must draw a meaningful picture.**

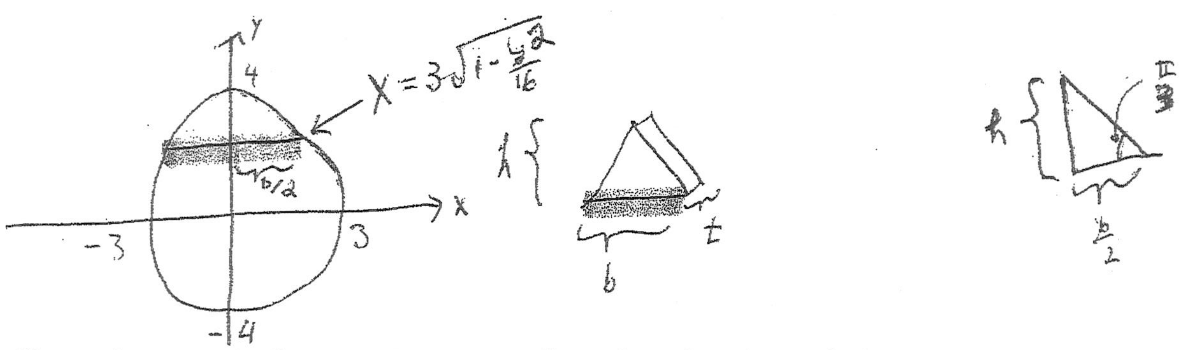


Chop the y-axis from $y=0$ to $y=4$. Consider the rectangle with y-coordinate y. Revolve the rectangle about $y=-1$. Get a shell of volume $2\pi rht$, with $t=dy$, $r=y+1$ and $h=1-((y-2)^2-3)=4y-y^2$. The volume of the shell is $2\pi rht=2\pi(y+1)(4y-y^2)dy$. The volume of the solid is

$$2\pi\int_0^4(-y^3+3y^2+4y)dy=2\pi(-y^4/4+y^3+2y^2)|_0^4=2\pi(-64+64+32)=\boxed{64\pi}.$$

4. (6 points) **Consider the solid whose base is bounded by $\frac{x^2}{9}+\frac{y^2}{16}=1$ in the xy-plane. Each cross section of the solid perpendicular to the y-axis and perpendicular to the base is an equilateral triangle. Find the volume of the solid. You must draw a meaningful picture.**

The base of the solid is an ellipse:



Chop the y-axis from -4 to 4. Consider the slice of the solid with y-coordinate y. This slice has volume $(1/2)bht$. We have $t=dy$, $\frac{h}{(b/2)}=\frac{\text{Op}}{\text{Adj}}=\tan\frac{\pi}{3}=\sqrt{3}$. So, $h=\sqrt{3}\frac{b}{2}$. We see that $\frac{b}{2}$ is the x-coordinate on the ellipse that corresponds to y; so, $\frac{b}{2}=3\sqrt{1-\frac{y^2}{16}}$. The volume of the slice is

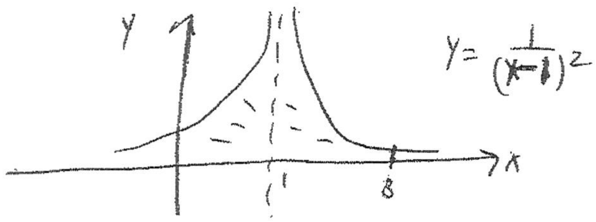
$$\tfrac{1}{2}bht=\sqrt{3}\tfrac{b}{2}\tfrac{b}{2}t=\sqrt{3}\left(\tfrac{b}{2}\right)^2t=9\sqrt{3}(1-\tfrac{y^2}{16})dy.$$

The volume of the solid is

$$9\sqrt{3}\int_{-4}^{4}\left(1-\frac{y^2}{16}\right)dy = 9\sqrt{3}\left(y-\frac{y^3}{48}\right)\Big|_{-4}^{4} = \boxed{18\sqrt{3}\left(4-\frac{4}{3}\right)}.$$

5. (6 points) **Find** $\int_0^3 \frac{1}{(x-1)^2}dx$.

The function $f(x)=\frac{1}{(x-1)^2}$ goes to infinity as x gets near 1:



The answer will either be a positive number or $+\infty$. The integral is equal to

$$\lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^2}dx + \lim_{a\to 1^+}\int_a^3 \frac{1}{(x-1)^2}dx$$

$$= \lim_{b\to 1^-} \frac{1}{-(x-1)}\Big|_0^b + \lim_{a\to 1^+}\frac{1}{-(x-1)}\Big|_a^3$$

$$= \lim_{b\to 1^-}\frac{1}{-(b-1)} - 1 + \lim_{a\to 1^+}\left(\frac{1}{-2}+\frac{1}{a-1}\right)$$

Both limits $\lim\limits_{b\to 1^-}\frac{1}{-(b-1)}$ and $\lim\limits_{a\to 1^+}\frac{1}{a-1}$ equal $+\infty$. The answer is $\boxed{+\infty}$.

6. (6 points) **Find** $\int \cos^5 x dx$. **You must check your answer.**

Save one $\cos x$. Let $u=\sin x$. It follows that $du=\cos x dx$. The integral is equal to

$$\int(1-\sin^2 x)^2\cos x dx = \int(1-u^2)^2 du = \int(1-2u^2+u^4)du = u-\frac{2}{3}u^3+\frac{u^5}{5}+C$$

$$= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C}.$$

Check. The derivative of the proposed answer is

$$\cos x - 2\sin^2 x\cos x + \sin^4 x\cos x = \cos x(1-2\sin^2 x+\sin^4 x) = \cos x(1-\sin^2 x)^2.$$

7. (7 points) **Find** $\int \sec x \tan^2 x dx$ **. You must check your answer.**

Find $\int \sec x \tan^2 x\, dx$ **. Check your answer.**

We do this integral by parts. Let $u = \tan x$ and $dv = \sec x \tan x dx$. We compute $du = \sec^2 x dx$ and $v = \sec x$. We have

$$\int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec^3 x dx = \sec x \tan x - \int \sec x(\tan^2 x + 1) dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x dx - \int \sec x dx.$$

Add $\int \sec x \tan^2 x dx$ to both sides

$$2 \int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec x dx = \sec x \tan x - \ln|\sec x + \tan x| + C.$$

Divide by 2 to see that

$$\boxed{\int \sec x \tan^2 x\, dx = \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C.}$$

Check . The derivative of the proposed answer is

$$\frac{1}{2}\left(\sec x \sec^2 x + \sec x \tan x \tan x - \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\right)$$

$$= \frac{1}{2}\left(\sec x \sec^2 x + \sec x \tan x \tan x - \sec x\right)$$

$$= \frac{1}{2} \sec x \left(\sec^2 x + \tan^2 x - 1\right) = \frac{1}{2} \sec x \left(\tan^2 x + \tan^2 x\right). \checkmark$$

8. (7 points) **Find** $\int \frac{4x^3+6x^2+4x+8}{x^4+4x^2} dx$ **. You must check your answer.**

We see that $x^4 + 4x^2 = x^2(x^2+4)$. Write

$$\frac{4x^3 + 6x^2 + 4x + 8}{x^4 + 4x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 4}.$$

Multiply both sides by $x^4 + 4x^2$ to get

$$4x^3 + 6x^2 + 4x + 8 = Ax(x^2 + 4) + B(x^2 + 4) + (Cx + D)x^2;$$

so,

$$4x^3 + 6x^2 + 4x + 8 = (A + C)x^3 + (B + D)x^2 + 4Ax + 4B.$$

Equate the corresponding coefficients to obtain $B = 2$, $A = 1$, $B + D = 6$ (hence $D = 4$), and $A + C = 4$ (so $C = 3$). We check that

$$\frac{1}{x} + \frac{2}{x^2} + \frac{3x + 4}{x^2 + 4} = \frac{x(x^2 + 4) + 2(x^2 + 4) + (3x + 4)x^2}{x^2(x^2 + 4)} = \frac{4x^3 + 6x^2 + 4x + 8}{x^2(x^2 + 4)}.$$

The original integral is equal to

$$\int (\tfrac{1}{x} + \tfrac{2}{x^2} + \tfrac{3x+4}{x^2+4})dx = \boxed{\ln|x| - \tfrac{2}{x} + \tfrac{3}{2}\ln|x^2 + 4| + 2\arctan\tfrac{x}{2} + C}$$