Math 142, Exam 2, Spring 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (9 points) **Find** $\int \frac{-x^2+3x+1}{x(x^2+1)}dx$ **. Please check your answer.**

We look for integers A, B, and C with

$$\frac{-x^2+3x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiply both sides by $x(x^2+1)$ to obtain

$$-x^2+3x+1 = A(x^2+1)+(Bx+C)x$$

$$-x^2+3x+1 = (A+B)x^2+Cx+A$$

Equate the corresponding coefficients to learn that $A+B=-1$, $C=3$, $A=1$. Drop $A=1$ into $A+B=-1$ to see that $B=-2$. We have shown that

$$\frac{-x^2+3x+1}{x(x^2+1)} = \frac{1}{x} + \frac{-2x+3}{x^2+1}.$$

We verify this much before continuing. The right side is

$$\frac{(x^2+1)+(-2x+3)x}{x(x^2+1)} = \frac{-x^2+3x+1}{x(x^2+1)}.$$

Now we compute

$$\int \frac{-x^2+3x+1}{x(x^2+1)}dx = \int \left(\frac{1}{x} + \frac{-2x+3}{x^2+1}\right) dx$$

$$= \boxed{\ln|x| - \ln|x^2+1| + 3\arctan x + C}$$

2. (9 points) **Find** $\displaystyle\int \frac{1}{x^2+6x+10}dx$ **. Please check your answer.**

$$\int \frac{1}{x^2+6x+10}dx = \int \frac{1}{(x+3)^2+1}dx = \boxed{\arctan(x+3)+C}$$

3. (8 points) **Find** $\displaystyle\int_{-1}^{3} \frac{1}{(x-2)^2}\,dx$ **.**

We drew a picture elsewhere. The picture shows that the answer is either a positive number or plus infinity.

$$\int_{-1}^{3} \frac{1}{(x-2)^2}\,dx = \lim_{b\to 2^-}\int_{-1}^{b} \frac{1}{(x-2)^2}\,dx + \lim_{a\to 2^+}\int_{a}^{3} \frac{1}{(x-2)^2}\,dx$$

$$= \lim_{b\to 2^-} \left.\frac{-1}{(x-2)}\right|_{-1}^{b} + \lim_{a\to 2^+} \left.\frac{-1}{(x-2)}\right|_{a}^{3}$$

$$= \lim_{b\to 2^-}\left(\frac{-1}{(b-2)} - \frac{-1}{(-1-2)}\right) + \lim_{a\to 2^+}\left(\frac{-1}{(3-2)} - \frac{-1}{(a-2)}\right)$$

$$= +\infty - \frac{1}{3} - 1 + \infty = \boxed{+\infty}$$

4. (8 points) **Find** $\displaystyle\int \sin 3x\ \cos 4x\,dx$ **.**

Add the identities

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

to obtain

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B.$$

It follows that

$$\frac{1}{2}[\sin(A+B)+\sin(A-B)] = \sin A\cos B$$

and

$$\int \sin 3x\ \cos 4x\,dx = \frac{1}{2}\int(\sin(7x) - \sin x)\,dx = \boxed{\frac{1}{2}\left(\frac{-\cos 7x}{7} + \cos x\right) + C}$$

5. (8 points) **Find the limit of the sequence whose n^{th} term is $a_n = (\frac{n}{n+3})^n$.**

We first do a different problem.

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n}{n+3}\right) = \lim_{n\to\infty} \frac{\ln(\frac{n}{n+3})}{\frac{1}{n}}.$$

The top and the bottom both go to zero; so we apply L'hopital's rule. Keep in mind that $\ln(\frac{n}{n+3}) = \ln n - \ln(n+3)$. We learn that

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n+3}}{-\frac{1}{n^2}} = \lim_{n\to\infty} -n^2\left(\frac{1}{n} - \frac{1}{n+3}\right) = \lim_{n\to\infty} -n^2\left(\frac{n+3-n}{n(n+3)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-3n}{n+3}\right) = \lim_{n\to\infty}\left(\frac{-3}{1+\frac{3}{n}}\right) = -3.$$

We are now ready to do the original problem:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = \boxed{e^{-3}.}$$

6. (8 points) **Write the repeating decimal $1.42\overline{973} = 1.42973973973\ldots$ as the ratio of two integers.**

Let $r = 1.42973973973\ldots$. We see that $1000r - r = 1428.31$. Thus, $999r = 1428.31$ and $r = \frac{1428.31}{999} = \boxed{\frac{142831}{99900}}$.