Math 142, Exam 2, Fall 2013

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Your work must be coherent, complete, and correct. $\boxed{CIRCLE}$ your answer.

No Calculators or Cell phones.

1. (9 points) **Find the area bounded by $x + y^2 = 0$ and $2y = x + 3$. You must draw a meaningful picture.**

The graph of $x+y^2 = 0$ is a parabola with vertex at the origin and opening to the left. The graph of $2y = x + 3$ is the line through $(0, \frac{3}{2})$ and $(-3, 0)$. These two curves intersect when $(2y-3)+y^2 = 0$; so $y^2+2y-3 = 0$ or $(y+3)(y-1) = 0$. The intersection occurs when $y = -3$ or $y = 1$. The points of intersection are $(-1, 1)$ and $(-9, -3)$. The picture is on a separate page. We chop the y-axis from -3 to 1. The area is

$$\int_{-3}^{1}(-y^2-(2y-3))dy = \int_{-3}^{1}(-y^2-2y+3)dy = \frac{-y^3}{3} - y^2 + 3y\Big|_{-3}^{1}$$

$$= \frac{-1}{3} - 1 + 3 - (9-9-9) = \boxed{\frac{32}{3}}.$$

2. (9 points) **Consider a solid S. The base of S is the triangular region in the xy plane with vertices $(0,0)$, $(1,0)$, and $(0,1)$. The cross-sections of S perpendicular to the x-axis are squares. Find the volume of S. You must draw a meaningful picture.**

Look at the picture that appears on the separate page. We chop the x-axis from 0 to 1. Over each little piece of the x-axis we have a thin slice of the solid. The slice with x-coordinate x has volume s^2t, where $s = 1-x$ and $t = dx$. Thus, this slice has volume $(1-x)^2dx$. The volume of the solid is

$$\int_0^1 (1-x)^2dx = -\left.\frac{(1-x)^3}{3}\right|_0^1 = \boxed{\frac{1}{3}}.$$

3. (8 points) **Consider the series $\sum\limits_{n=2}^{\infty} \ln \frac{n}{n+2}$.**

(a) **Find a closed formula for the partial sum** $s_N = \sum_{n=2}^{N} \ln \frac{n}{n+2}$ **. (Recall that a closed formula is a formula which does not have any summation signs or any dots.)**

We see that

$$s_N = \ln \tfrac{2}{4} + \ln \tfrac{3}{5} + \ln \tfrac{4}{6} + \ln \tfrac{5}{7} + \ln \tfrac{6}{8} + \cdots + \ln \tfrac{N-3}{N-1} + \ln \tfrac{N-2}{N} + \ln \tfrac{N-1}{N+1} + \ln \tfrac{N}{N+2}$$

$$\begin{aligned}=&(\ln 2 - \underline{\ln 4}) + (\ln 3 - \underline{\ln 5}) + (\underline{\ln 4} - \underline{\ln 6}) + (\underline{\ln 5} - \underline{\ln 7}) + (\underline{\ln 6} - \underline{\ln 8})\\ &+ \cdots + (\underline{\ln(N-3)} - \underline{\ln(N-1)}) + (\underline{\ln(N-2)} - \underline{\ln N}) + (\underline{\ln(N-1)} - \ln(N+1))\\ &+ (\underline{\ln N} - \ln(N+2))\end{aligned}$$

$$\boxed{\ln 2 + \ln 3 - \ln(N+1) - \ln(N+2)}$$

(b) **Find the sum of the series** $\sum_{n=2}^{\infty} \ln \frac{n}{n+2}$ **. Justify your answer. Write in complete sentences.**

The sum of a series is the limit of the sequence of partial sums; so

$$\sum_{n=2}^{\infty} \ln \frac{n}{n+2} = \lim_{N\to\infty} s_N = \lim_{N\to\infty} (\ln 2 + \ln 3 - \ln(N+1) - \ln(N+2)) = -\infty.$$

So,

$$\boxed{\text{the series } \sum_{n=2}^{\infty} \ln \frac{n}{n+2} \text{ diverges to } -\infty.}$$

4. (8 points) **Approximate** $\sum_{n=1}^{\infty} \frac{1}{n^6}$ **with an error of at most** $\frac{5}{10^5}$ **. Justify your answer. Write in complete sentences.**

My answer refers to the picture which appears on a separate page. We use $\sum_{n=1}^{N} \frac{1}{n^6}$ to approximate $\sum_{n=1}^{\infty} \frac{1}{n^6}$ for some carefully chosen N. We see that the distance between $\sum_{n=1}^{N} \frac{1}{n^6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^6}$ is

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^6} - \sum_{n=1}^{N} \frac{1}{n^6} \right| = \sum_{n=N+1}^{\infty} \frac{1}{n^6} = \text{the area inside the boxes} \le$$

$$\text{the area under the curve} = \int_N^{\infty} \frac{1}{x^6} dx = \lim_{b\to\infty} \frac{1}{-5x^5}\bigg|_N^b = \lim_{b\to\infty} \left(\frac{1}{-5b^5} + \frac{1}{5N^5}\right) = \frac{1}{5N^5}.$$

We want the distance between $\sum_{n=1}^{N} \frac{1}{n^6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^6}$ to be at most $\frac{5}{10^5}$. We know that the distance between $\sum_{n=1}^{N} \frac{1}{n^6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^6}$ is at most $\frac{1}{5N^5}$. So, we pick N large enough that $\frac{1}{5N^5} \le \frac{5}{10^5}$. So we pick N large enough that $10^5 \le 25N^5$. It is clear that when $N = 10$, then $10^5 \le 25N^5$. We conclude that

$$\boxed{\sum_{n=1}^{10} \frac{1}{n^6} \text{ approximates } \sum_{n=1}^{\infty} \frac{1}{n^6} \text{ with an error at most } \frac{5}{10^5}.}$$

5. (8 points) **Does the series $\sum_{n=1}^{\infty} \frac{2n^2+n}{n^3}$ converge? Justify your answer. Write in complete sentences.**

We compare the given series to $\sum_{n=1}^{\infty} \frac{2}{n}$, which is twice the harmonic series. The harmonic series diverges; so twice the harmonic series diverges; furthermore, we see that

$$\frac{2}{n} < \frac{2+\frac{1}{n}}{n} = \frac{2n^2+n}{n^3}.$$

We apply the comparison test. That is, both series $\sum_{n=1}^{\infty} \frac{2n^2+n}{n^3}$ and $\sum_{n=1}^{\infty} \frac{2}{n}$ are series of positive numbers, $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges; each term of $\sum_{n=1}^{\infty} \frac{2n^2+n}{n^3}$ is greater than the corresponding term of $\sum_{n=1}^{\infty} \frac{2}{n}$; hence,

$$\boxed{\sum_{n=1}^{\infty} \frac{2n^2+n}{n^3} \text{ also diverges.}}$$

6. (8 points) **Does the series $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ converge? Justify your answer. Write in complete sentences.**

We apply the ratio test. We compute

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n-1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{10^n}{(n+1)4^{2n+1}}}{\frac{10^{n-1}}{(n)4^{2(n-1)+1}}} \right| = \lim_{n\to\infty} \frac{10^n}{(n+1)4^{2n+1}} \frac{(n)4^{2n-1}}{10^{n-1}}$$

$$= \lim_{n\to\infty} \frac{10n}{16(n+1)} = \lim_{n\to\infty} \frac{10}{16(1+\frac{1}{n})} = \frac{10}{16} < 1.$$

Thus $\rho < 1$ and

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}} \text{ converges}}$$

by the ratio test.