Math 142, Exam 2, Fall 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. This work must be coherent and correct. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

The solutions will be posted later today.

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a,b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum\limits_{i=1}^{n} f(x_i^*)\Delta_i$.

2. (6 points) **Find** $\displaystyle\int \frac{e^{3x}}{\sqrt{e^{3x}+1}}\,dx$. **Check your answer.**

Let $u = e^{3x}+1$. It follows that $du = 3e^{3x}dx$. The original problem is equal to

$$\frac{1}{3}\int u^{-1/2}du = \frac{1}{3}(2)u^{1/2} + C = \boxed{\frac{2}{3}\sqrt{e^{3x}+1} + C.}$$

Check: The derivative of the proposed answer is

$$\frac{2}{3}\frac{1}{2}(e^{3x}+1)^{-1/2}3e^{3x}. \checkmark$$

3. (6 points) **Find** $\displaystyle\int \frac{x^2-3}{(x-1)^3}\,dx$. **Check your answer.**

We find numbers A, B, and C with

$$\frac{x^2-3}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}.$$

Multiply both sides by $(x-1)^3$ to get

$$x^2-3=A(x-1)^2+B(x-1)+C$$
$$x^2-3=x^2A-2xA+A+Bx-B+C$$
$$x^2-3=x^2A+x(-2A+B)+A-B+C.$$

Equate the corresponding coefficients to see that $1=A$, $0=-2A+B$ (so $2=B$) and $-3=A-B+C$, (so $-3-1+2=C$; that is, $-2=C$.) We check:

$$\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{-2}{(x-1)^3}=\frac{x^2-2x+1+2x-2-2}{(x-1)^3}=\frac{x^2-3}{(x-1)^3},$$

as desired. So, the original problem is equal to

$$\int \frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{-2}{(x-1)^3}\,dx=\boxed{\ln|x-1|-\frac{2}{x-1}+\frac{1}{(x-1)^2}+C}$$

Check: The derivative of the proposed answer is

$$\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{-2}{(x-1)^3},$$

and we already saw that this last expression is equal to $\frac{x^2-3}{(x-1)^3}$.

4. (6 points) **Find** $\int \frac{1}{x^2-3x+2}\,dx$. **Check your answer.**

The denominator factors as $(x-2)(x-1)$. We find numbers A and B with

$$\frac{1}{x^2-3x+2}=\frac{A}{x-2}+\frac{B}{x-1}.$$

Multiply both sides by $(x-1)(x-2)$ to get

$$1=A(x-1)+B(x-2).$$

Plug in $x=1$ to see that $B=-1$. Plug in $x=2$ to see that $1=A$. We check that

$$\frac{1}{x-2}+\frac{-1}{x-1}=\frac{x-1-(x-2)}{(x-1)(x-2)}=\frac{1}{(x-1)(x-2)}.$$

So, the original problem is equal to

$$\int \frac{1}{x-2}+\frac{-1}{x-1}dx=\boxed{\ln|x-2|-\ln|x-1|+C}.$$

Check: The derivative of the proposed answer is $\frac{1}{x-2}+\frac{-1}{x-1}$ and we already saw that this last expression is equal to $\frac{1}{x^2-3x+2}$.

5. (6 points) **Find** $\displaystyle\int \frac{1}{4x^2+8x+5}\,dx$ **. Check your answer.**

The denominator does not factor. We complete the square

$$\int \frac{1}{4x^2+8x+5}\,dx = \int \frac{1}{4(x^2+2x+\boxed{1})+5-4(\boxed{1})}\,dx = \int \frac{1}{4(x+1)^2+1}\,dx.$$

Let $u = 2(x+1)$. It follows that $du = 2dx$. This integral is

$$= \frac{1}{2}\int \frac{du}{u^2+1}\,dx = \frac{1}{2}\arctan u + C = \boxed{\frac{1}{2}\arctan(2x+2)+C}.$$

Check: The derivative of the proposed answer is

$$\frac{1}{2}\frac{2}{(2x+2)^2+1} = \frac{1}{4x^2+8x+5}.$$

6. (6 points) **Find** $\displaystyle\int_{-1}^{4} \frac{dx}{(x-1)^2}$ **.**

The picture appears elsewhere. The function $f(x) = \frac{1}{(x-1)^2}$ goes to infinity as x goes to 1. Furthermore $f(x)$ is always positive. This integral is improper and the answer is either a finite positive number or $+\infty$.

$$\int_{-1}^{4} \frac{dx}{(x-1)^2}$$

$$= \lim_{b\to 1^-}\int_{-1}^{b} \frac{dx}{(x-1)^2} + \lim_{a\to 1^+}\int_{a}^{4} \frac{dx}{(x-1)^2}$$

$$= \lim_{b\to 1^-} \left.\frac{-1}{(x-1)}\right|_{-1}^{b} + \lim_{a\to 1^+} \left.\frac{-1}{(x-1)}\right|_{a}^{4}$$

$$= \lim_{b\to 1^-} \frac{-1}{(b-1)} - \frac{-1}{(-1-1)} + \lim_{a\to 1^+} \frac{-1}{(4-1)} - \frac{-1}{(a-1)}$$

$$= +\infty - \frac{1}{2} + \frac{1}{3} + \infty.$$

$$\boxed{\text{The integral diverges to } +\infty.}$$

7. (7 points)**Find the volume of the solid that is obtained by revolving the region bounded by $y = x^2$ and $y - x = 2$ about the line $x = 5$. You must draw a meaningful picture.**

The picture appears elsewhere. The intersection points are found by solving $x^2 - x - 2 = 0$ and this is $(x-2)(x+1)$. So $x = 2$ and $x = -1$. The intersection points are $(-1, 1)$ and $(2, 4)$. Notice that these points satisfy both equations $y = x^2$ and $y - x = 2$. Chop the x-axis from $x = -1$ to $x = 2$. Consider the rectangle with x-coordinate x. Revolve this rectangle about the line $x = 5$ to obtain a shell of volume $2\pi rht$, where $t = dx$, $r = 5 - x$, and $h = x + 2 - x^2$. The volume of the shell is $2\pi rht = 2\pi(5-x)(x+2-x^2)dx$. The volume of the solid is

$$2\pi \int_{-1}^{2} (5-x)(x+2-x^2)dx = 2\pi \int_{-1}^{2} (5x + 10 - 5x^2 - x^2 - 2x + x^3)dx$$

$$= 2\pi \int_{-1}^{2} (3x + 10 - 6x^2 + x^3)dx$$

$$= 2\pi\left(\tfrac{3}{2}x^2 + 10x - 2x^3 + \tfrac{1}{4}x^4\Big|_{-1}^{2}\right) = \boxed{2\pi\left(\left(\tfrac{3}{2}2^2 + 20 - 16 + 4\right) - \left(\tfrac{3}{2} - 10 + 2 + \tfrac{1}{4}\right)\right)}$$

8. (7 points) **Consider a solid S whose base in the xy plane is the region bounded by $y = x^2$ and $y - x = 2$. Each cross-section of S perpendicular to the x-axis is an equilateral triangle. Find the volume of S. You must draw a meaningful picture.**

The picture appears elsewhere. The intersection points are still $(-1, 1)$ and $(2, 4)$. Chop the x-axis from $x = -1$ to $x = 2$. Consider the slice of S with x-coordinate x. This slice is an equilateral triangle with thickness. The volume of the slice is the area of the triangle times the thickness and this is $\frac{1}{2}bht$, where $t = dx$ $b = x + 2 - x^2$ and $h = \frac{\sqrt{3}}{2}b$. So the volume of the slice is

$$\tfrac{1}{2}bht = \tfrac{1}{2}b\tfrac{\sqrt{3}}{2}b = \tfrac{\sqrt{3}}{4}b^2t = \tfrac{\sqrt{3}}{4}(x+2-x^2)^2dx.$$

The volume of the solid is

$$\tfrac{\sqrt{3}}{4}\int_{-1}^{2}(x+2-x^2)^2dx = \tfrac{\sqrt{3}}{4}\int_{-1}^{2}(x^2 + 4x - 2x^3 + 4 - 4x^2 + x^4)dx$$

$$= \tfrac{\sqrt{3}}{4}\int_{-1}^{2}(4x - 2x^3 + 4 - 3x^2 + x^4)dx = \tfrac{\sqrt{3}}{4}\left(2x^2 - \tfrac{1}{2}x^4 + 4x - x^3 + \tfrac{1}{5}x^5\Big|_{-1}^{2}\right)$$

$$= \boxed{\tfrac{\sqrt{3}}{4}\left(\left(8 - \tfrac{1}{2}2^4 + 8 - 8 + \tfrac{1}{5}2^5\right) - \left(2 - \tfrac{1}{2} - 4 + 1 - \tfrac{1}{5}\right)\right)}$$