Math 142, Exam 1, Spring 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Find** $\int \frac{dx}{(x^2-1)^{3/2}}$ **. Please check your answer.**

Answer: Let $x = \sec\theta$. It follows that $dx = \sec\theta\tan\theta d\theta$ and $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$. The original integral is

$$\int \frac{\sec\theta\tan\theta d\theta}{\tan^3\theta} = \int \frac{\cos\theta}{\sin^2\theta} d\theta = \int \csc\theta\cot\theta d\theta = -\csc\theta + C.$$

Draw a right triangle with $x = \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$. So this triangle has x on the hypotenuse, 1 on the adjacent, and $\sqrt{x^2-1}$ on the opposite. So $\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{x}{\sqrt{x^2-1}}$. Our integral is equal to

$$-\csc\theta + C = \boxed{\frac{-x}{\sqrt{x^2-1}} + C}.$$

Check: The derivative of the proposed answer is

$$-x(-1/2)(x^2-1)^{-3/2}(2x) - (x^2-1)^{-1/2} = (x^2-1)^{-3/2}(x^2-(x^2-1)) = (x^2-1)^{-3/2}. \checkmark$$

2. **Find** $\int \sec^4 x\,dx$ **. Please check your answer.**

Answer: We have to intgegrate an even power of $\sec x$ times some power of $\tan x$. We save $\sec^2 x$, convert all remaining $\sec x$'s into $\tan x$'s (using $\sec^2 x = \tan^2 x + 1$) and let $u = \tan x$ (and so $du = \sec^2 x dx$). So

$$\int \sec^4 x dx = \int (\tan^2 x + 1)\sec^2 x dx = \int (u^2+1) du = \frac{u^3}{3} + u + C = \boxed{\frac{\tan^3 x}{3} + \tan x + C}$$

Check: The derivative of the proposed answer is

$$\tan^2 x \sec^2 x + \sec^2 x = \sec^2 x(\tan^2 x + 1) = \sec^4 x. \checkmark$$

3. **Find** $\int \cos 3x \, \cos 4x \, dx$ **.**

Answer: Add

$$\begin{aligned}\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi\\ \cos(\theta-\phi) &= \cos\theta\cos\phi + \sin\theta\sin\phi\end{aligned}$$

to obtain

$$\cos(\theta+\phi) + \cos(\theta-\phi) = 2\cos\theta\cos\phi;$$

hence,

$$\frac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)] = \cos\theta\cos\phi.$$

Take $\theta = 3x$ and $\phi = 4x$ to see that

$$\int \cos 3x \, \cos 4x \, dx = \frac{1}{2}\int \cos(7x) + \cos(x) dx = \boxed{\frac{1}{2}\left[\frac{\sin 7x}{7} + \sin x\right] + C}$$

4. **Find** $\int e^{4x} \sin x dx$ **. Please check your answer.**

Answer: Apply integration by parts. Let $u = e^{4x}$ and $dv = \sin x dx$. Calculate $du = 4e^{4x} dx$ and $v = -\cos x$. We see that

$$\int e^{4x} \sin x dx = -e^{4x}\cos x + 4\int e^{4x}\cos x dx.$$

Let $u = e^{4x}$ and $dv = \cos x dx$. Calculate $du = 4e^{4x}dx$ and $v = \sin x$. We see that

$$\int e^{4x}\sin x dx = -e^{4x}\cos x + 4\left[e^{4x}\sin x - 4\int e^{4x}\sin x dx\right].$$

Add $16\int e^{4x}\sin x dx$ to both sides to obtain

$$17\int e^{4x}\sin x dx = -e^{4x}\cos x + 4e^{4x}\sin x + C.$$

It follows that

$$\int e^{4x}\sin x dx = \frac{1}{17}[-e^{4x}\cos x + 4e^{4x}\sin x] + K = \boxed{\frac{e^{4x}}{17}[-\cos x + 4\sin x] + K}.$$

Check: The derivative of the proposed answer is

$$\frac{e^{4x}}{17}[\sin x + 4\cos x + 4(-\cos x + 4\sin x)] = e^{4x}\sin x. \checkmark$$

5. **Find the area of the region bounded by** $y^2 - x - 1 = 0$ **and** $y - x + 1 = 0$**. Please draw a meaningful picture.**

Answer: The picture is on a separate page. Observe that $y^2 - 1 = x$ is a parabola, on its side shifted one unit to the left; and $y + 1 = x$ is a line. The intersection points occur when $y^2 - (y+1) - 1 = 0$. This occurs when $y^2 - y - 2 = 0$, or $(y-2)(y+1) = 0$; so $y = 2$ or $y = -1$. The intersection points are $(3, 2)$ and $(0, -1)$. We integrate from bottom to top of right minus left. The area is

$$\int_{-1}^{2} [(y+1) - (y^2 - 1)]dy = \int_{-1}^{2} (y + 2 - y^2)dy = \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_{-1}^{2}$$

$$= \boxed{2 + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)}.$$