Math 142, Exam 1, Spring 2011

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

There are 9 problems. The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (5 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a, b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum_{i=1}^{n} f(x_i^*)\Delta_i$.

2. (5 points) **State both parts of the Fundamental Theorem of Calculus. Be sure to explain all of your notation. Write in complete sentences.**

Let f be a continuous function defined on the closed interval $[a, b]$.

(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a, b]$, then $A'(x) = f(x)$ for all $x \in [a, b]$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. (5 points) **Find** $\int \sqrt{9 - 4x^2}\, dx$. **Check your answer.**

Let $2x = 3\sin\theta$. Calculate $2dx = 3\cos\theta d\theta$. Calculate

$$\sqrt{9 - 4x^2} = \sqrt{9 - 9\sin^2\theta} = 3\sqrt{1 - \sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta.$$

The original integral is equal to

$$\tfrac{9}{2}\int \cos^2\theta d\theta = \tfrac{9}{4}\int((1 + \cos(2\theta))d\theta = \tfrac{9}{4}(\theta + \tfrac{1}{2}\sin(2\theta)) + C$$

$$= \tfrac{9}{4}(\theta + \sin\theta\cos\theta) + C = \tfrac{9}{4}(\arcsin(\tfrac{2x}{3}) + \tfrac{2x}{3}\tfrac{\sqrt{9-4x^2}}{3}) + C$$

$$\boxed{= \tfrac{9}{4}\arcsin(\tfrac{2x}{3}) + \tfrac{x\sqrt{9-4x^2}}{2} + C}$$

<u>Check</u>. The derivative of the proposed answer is

$$\frac{9}{4}\frac{\frac{2}{3}}{\sqrt{1-\left(\frac{2x}{3}\right)^2}}+\frac{1}{2}\left[x\frac{-8x}{2\sqrt{9-4x^2}}+\sqrt{9-4x^2}\right]$$

$$=\frac{3}{2\sqrt{(\frac{1}{9})(9-4x^2)}}+\frac{-4x^2}{2\sqrt{9-4x^2}}+\frac{1}{2}\sqrt{9-4x^2}$$

$$=\frac{9}{2\sqrt{9-4x^2}}+\frac{-4x^2}{2\sqrt{9-4x^2}}+\frac{1}{2}\sqrt{9-4x^2}$$

$$=\frac{1}{2}\frac{9-4x^2}{\sqrt{9-4x^2}}+\frac{1}{2}\sqrt{9-4x^2}$$

$$=\frac{1}{2}\left[\sqrt{9-4x^2}+\sqrt{9-4x^2}\right]=\sqrt{9-4x^2}.\ \checkmark$$

4. (5 points) **Find** $\int \cos 3x \cos 4x\, dx$ **.**

You know that

$$\cos(A+B)=\cos A\cos B-\sin A\sin B \quad \text{and} \quad \cos(A-B)=\cos A\cos B+\sin A\sin B.$$

Add these identities to get

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B;$$

which is the same as

$$\tfrac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]=\cos A\cos B.$$

Our integral is

$$\tfrac{1}{2}\int(\cos 7x+\cos x)\,dx=\boxed{\tfrac{1}{2}(\tfrac{1}{7}\sin(7x)+\sin x)+C}.$$

5. (6 points)**Find** $\int e^{3x}\sin x\, dx$ **. Check your answer.**

Use integration by parts. Let $u=e^{3x}$ and $dv=\sin x dx$. It follows that $du=3e^{3x}dx$ and $v=-\cos x$. We have:

$$\int e^{3x}\sin x dx=-e^{3x}\cos x+3\int e^{3x}\cos x dx.$$

Let $u = e^{3x}$ and $dv = \cos x dx$. It follows that $du = 3e^{3x}dx$ and $v = \sin x$. We have:

$$\int e^{3x} \sin x dx = -e^{3x} \cos x + 3 \int e^{3x} \cos x dx$$

$$= -e^{3x} \cos x + 3 \left(e^{3x} \sin x - 3 \int e^{3x} \sin x dx \right).$$

So,

$$\int e^{3x} \sin x dx = -e^{3x} \cos x + 3e^{3x} \sin x - 9 \int e^{3x} \sin x dx.$$

Add $9 \int e^{3x} \sin x dx$ to both sides of the equation to get

$$10 \int e^{3x} \sin x dx = -e^{3x} \cos x + 3e^{3x} \sin x.$$

Divide by 10; don't forget to add $+C$. We have

$$\boxed{\int e^{3x} \sin x dx = \tfrac{1}{10} \left(-e^{3x} \cos x + 3e^{3x} \sin x\right) + C}.$$

<u>Check</u>. The derivative of the proposed answer is

$$\frac{1}{10} \left(-e^{3x}(-\sin x) - 3e^{3x} \cos x + 3e^{3x} \cos x + 9e^{3x} \sin x\right) = e^{3x} \sin x. \checkmark$$

6. (6 points) **Find** $\int \sin^7 x \, dx$ **. Check your answer.**

Save one $\sin x$. Convert the rest to $\cos x$. Let $u = \cos x$. It follows that $du = -\sin x$. Our integral equals

$$\int (1 - \cos^2 x)^3 \sin x dx = -\int (1 - u^2)^3 du = -\int (1 - 3u^2 + 3u^4 - u^6) du$$

$$= -(u - u^3 + \tfrac{3}{5} u^5 - \tfrac{u^7}{7}) + C$$

$$= \boxed{-(\cos x - \cos^3 x + \tfrac{3}{5} \cos^5 x - \tfrac{\cos^7 x}{7}) + C}.$$

<u>Check</u>. The derivative of the proposed answer is

$$-(-\sin x - 3\cos^2 x(-\sin x) + 3\cos^4 x(-\sin x) - \cos^6 x(-\sin x))$$

$$= \sin x(1 - 3\cos^2 x + 3\cos^4 x - \cos^6 x) = \sin x(1 - \cos^2 x)^3. \checkmark$$

7. (6 points) **Find** $\int \tan^7 x\,dx$ **. Check your answer.**

Save $\tan x$, convert the rest of the tangents to secents. Our integral is equal to

$$\int (\sec^2 x - 1)^3 \tan x\,dx = \int (\sec^6 x - 3\sec^4 x + 3\sec x^2 x - 1)\tan x\,dx$$

$$= \int (\sec^6 x - 3\sec^4 x + 3\sec x^2 x)\tan x\,dx + \int \frac{-\sin x}{\cos x} dx.$$

In the first integral, let $u = \sec x$. It follows that $du = \sec x \tan x dx$. Our integral is equal to

$$\int (u^5 - 3u^3 + 3u)du + \ln|\cos x| + C = \tfrac{u^6}{6} - \tfrac{3u^4}{4} + \tfrac{3u^2}{2} + \ln|\cos x| + C$$

$$= \boxed{\tfrac{\sec^6 x}{6} - \tfrac{3\sec^4 x}{4} + \tfrac{3\sec^2 x}{2} + \ln|\cos x| + C}.$$

Check. The derivative of the proposed answer is

$$\sec^5 x(\sec x \tan x) - 3\sec^3 x(\sec x \tan x) + 3\sec x(\sec x \tan x) - \tan x$$

$$= \tan x(\sec^6 x - 3\sec^4 x + 3\sec^2 x - 1) = \tan x(\sec^2 x - 1)^3 \checkmark.$$

8. (6 points) **Find** $\int \cos^2 x\,dx$ **.**

Our integral is $\frac{1}{2}\int (1 + \cos 2x)dx = \boxed{\frac{1}{2}(x + \frac{\sin 2x}{2}) + C}$.

9. (6 points) **Find** $\int \sin x\sqrt{\cos^2 x + 16}\,dx$ **. Check your answer.**

Let $w = \cos x$. It follows that $dw = -\sin x dx$. Our integral is $-\int \sqrt{w^2 + 16}dw$. Let $w = 4\tan\theta$. It follows that

$$dw = 4\sec^2\theta d\theta \quad \text{and} \quad \sqrt{w^2 + 16} = \sqrt{16\tan^2\theta + 16} = 4\sqrt{\sec^2\theta} = 4\sec\theta.$$

Our integral is $-16\int \sec^3\theta d\theta$. We integrate $\sec^3\theta$ off on the side. Let $u = \sec\theta$ and $dv = \sec^2\theta d\theta$. It follows that

$$du = \sec\theta\tan\theta d\theta \quad \text{and} \quad v = \tan\theta;$$

hence

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta.$$

Add $\int \sec^3\theta d\theta$ to both sides and divide by 2 to see that

$$\int \sec^3\theta d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\,\right] + C.$$

Our integral is

$$= -8\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right] + C$$

$$= -8\left[\frac{\sqrt{w^2+16}}{4}\frac{w}{4} + \ln\left|\frac{\sqrt{w^2+16}}{4} + \frac{w}{4}\right|\right] + C$$

$$= -8\left[\frac{\sqrt{\cos^2 x+16}}{4}\frac{\cos x}{4} + \ln\left|\frac{\sqrt{\cos^2 x+16}}{4} + \frac{\cos x}{4}\right|\right] + C$$

$$= \frac{-\cos x\sqrt{\cos^2 x+16}}{2} - 8\left(\ln\left|\sqrt{\cos^2 x+16}+\cos x\right| - \ln 4\right) + C$$

Let K be the constant $8\ln 4 + C$. Our answer is

$$= \boxed{\frac{-\cos x\sqrt{\cos^2 x+16}}{2} - 8\ln\left|\sqrt{\cos^2 x+16}+\cos x\right| + K}$$

Check. The derivative of the proposed answer is

$$-\frac{1}{2}\left(\frac{\cos x(-2\cos x\sin x)}{2\sqrt{\cos^2 x+16}} - \sin x\sqrt{\cos^2 x+16}\right) - 8\frac{\frac{-2\cos x\sin x}{2\sqrt{\cos^2 x+16}} - \sin x}{\sqrt{\cos^2 x+16}+\cos x}$$

$$= -\frac{1}{2}(-\sin x)\left(\frac{\cos^2 x}{\sqrt{\cos^2 x+16}} + \sqrt{\cos^2 x+16}\right) - 8(-\sin x)\frac{\frac{\cos x}{\sqrt{\cos^2 x+16}} + 1}{\sqrt{\cos^2 x+16}+\cos x}$$

$$= \frac{1}{2}\sin x\left(\frac{\cos^2 x}{\sqrt{\cos^2 x+16}} + \sqrt{\cos^2 x+16}\right) + 8\sin x\frac{\cos x+\sqrt{\cos^2 x+16}}{(\sqrt{\cos^2 x+16}+\cos x)\sqrt{\cos^2 x+16}}$$

$$= \frac{\sin x}{2\sqrt{\cos^2 x+16}}\left(\cos^2 x+\cos^2 x+16+16\right) = \frac{\sin x}{\sqrt{\cos^2 x+16}}\left(\cos^2 x+16\right) = \sin x\sqrt{\cos^2 x+16}. \checkmark$$