Math 142, Exam 1, SOLUTION, Fall 2011

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Answer: Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a,b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum\limits_{i=1}^{n} f(x_i^*)\Delta_i$.

2. (6 points) **State both parts of the Fundamental Theorem of Calculus. Be sure to explain all of your notation. Write in complete sentences.**

Answer: Let f be a continuous function defined on the closed interval $[a,b]$.

(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a,b]$, then $A'(x) = f(x)$ for all $x \in [a,b]$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. (6 points) **Find** $\int \sin^2 x dx$ **.**

Answer: The given integral is equal to

$$\tfrac{1}{2}\int (1 - \cos 2x)dx = \boxed{\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C}.$$

4. (6 points) **Find** $\int \sin^5 x dx$ **. Check your answer.**

Answer: Save one $\sin x$ and convert the rest of the $\sin x$'s to $\cos x$. The given integral is equal to $\int (1 - \cos^2 x)^2 \sin x dx$. Let $u = \cos x$. It follows that $du = -\sin x dx$. The original integral is equal to

$$-\int (1 - u^2)^2 du = -\int (1 - 2u^2 + u^4) du = -(u - \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5) + C$$

$$= \boxed{-\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C}$$

Check . The derivative of the proposed answer is

$$\sin x + 2\cos^2 x(-\sin x) - \cos^4 x(-\sin x) = \sin x(1 - 2\cos^2 x + \cos^4 x)$$

$$= \sin x(1-\cos^2 x)^2 = \sin^5 x. \checkmark$$

5. (6 points) **Find** $\int e^{3x}\sin x dx$ **. Check your answer.**

Answer: Use integration by parts. Let $u = e^{3x}$ and $dv = \sin x dx$. It follows that $du = 3e^{3x}dx$ and $v = -\cos x$. We have:

$$\int e^{3x}\sin x dx = -e^{3x}\cos x + 3\int e^{3x}\cos x dx.$$

Let $u = e^{3x}$ and $dv = \cos x dx$. It follows that $du = 3e^{3x}dx$ and $v = \sin x$. We have:

$$\int e^{3x}\sin x dx = -e^{3x}\cos x + 3\int e^{3x}\cos x dx$$

$$= -e^{3x}\cos x + 3\left(e^{3x}\sin x - 3\int e^{3x}\sin x dx\right).$$

So,

$$\int e^{3x}\sin x dx = -e^{3x}\cos x + 3e^{3x}\sin x - 9\int e^{3x}\sin x dx.$$

Add $9\int e^{3x}\sin x dx$ to both sides of the equation to get

$$10\int e^{3x}\sin x dx = -e^{3x}\cos x + 3e^{3x}\sin x.$$

Divide by 10 ; don't forget to add $+C$. We have

$$\boxed{\int e^{3x}\sin x dx = \tfrac{1}{10}\left(-e^{3x}\cos x + 3e^{3x}\sin x\right) + C}.$$

Check . The derivative of the proposed answer is

$$\frac{1}{10}\left(-e^{3x}(-\sin x) - 3e^{3x}\cos x + 3e^{3x}\cos x + 9e^{3x}\sin x\right) = e^{3x}\sin x. \checkmark$$

6. (5 points) **Find** $\int \frac{dx}{x^2-5x+6}$ **. Check your answer.**

Answer: We see that $x^2 - 5x + 6 = (x-2)(x-3)$. We use the technique of partial fractions:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

Multiply both sides by $(x-2)(x-3)$:

$$1 = A(x-3) + B(x-2).$$

Plug in $x = 3$ to learn that $B = 1$. Plug in $x = 2$ to learn that $A = -1$. Verify that

$$\frac{-1}{(x-2)} + \frac{1}{(x-3)} = \frac{-(x-3)+(x-2)}{(x-2)(x-3)} = \frac{1}{(x-2)(x-3)}.$$

The original integral is

$$\int \left(\frac{-1}{(x-2)} + \frac{1}{(x-3)} \right) dx = \boxed{-\ln|x-2| + \ln|x-3| + C.}$$

<u>Check</u> . The derivative of the proposed answer is

$$\frac{-1}{(x-2)} + \frac{1}{(x-3)} = \frac{1}{(x-2)(x-3)}. \checkmark$$

7. (5 points) **Find** $\int \frac{dx}{4x^2+4x+10}$ **. Check your answer.**

Answer: The denominator does not factor; so we see that

$$4x^2 + 4x + 10 = (2x+1)^2 + 9.$$

Let $2x + 1 = 3\tan\theta$. We compute

$$4x^2 + 4x + 10 = (2x+1)^2 + 1 = (3\tan\theta)^2 + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta.$$

We also compute $2dx = 3\sec^2\theta d\theta$. The integral is

$$\frac{1}{2}\int \frac{3\sec^2\theta d\theta}{9\sec^2\theta} = \frac{1}{6}\int 1 d\theta = \frac{1}{6}\theta + C = \boxed{\frac{1}{6}\arctan\left(\frac{2x+1}{3}\right) + C}$$

Check. The derivative of the proposed answer is

$$\left(\frac{1}{6}\right)\left(\frac{\frac{2}{3}}{\left(\frac{2x+1}{3}\right)^2+1}\right)=\frac{1}{9\left[\left(\frac{2x+1}{3}\right)^2+1\right]}=\frac{1}{4x^2+4x+1+9}. \checkmark$$

8. (5 points) **Find** $\int \frac{e^{1/x}}{x^2}dx$ **. Check your answer.**

Answer: Let $u=\frac{1}{x}$. It follows that $du=\frac{-dx}{x^2}$. The integral is

$$-\int e^u du=-e^u+C=\boxed{-e^{\frac{1}{x}}.}$$

Check. The derivative of the proposed answer is

$$-e^{\frac{1}{x}}\left(\frac{-1}{x^2}\right). \checkmark$$

9. (5 points) **Find** $\int \sqrt{1-x^2}dx$ **. Check your answer.** Let $x=\sin\theta$. It follows that $dx=\cos\theta d\theta$. One computes that

$$\sqrt{1-x^2}=\sqrt{1-\sin^2 x}=\sqrt{\cos^2 x}=\cos x;$$

hence, the integral is

$$\int \cos^2\theta d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta=\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+C$$

$$=\frac{1}{2}\left(\theta+\frac{2\sin\theta\cos\theta}{2}\right)+C=\frac{1}{2}(\theta+\sin\theta\cos\theta)+C$$

$$=\boxed{\frac{1}{2}\left(\arcsin x+x\sqrt{1-x^2}\right)+C}.$$

Answer:

Check. The derivative of the proposed answer is

$$\frac{1}{2}\left(\frac{1}{\sqrt{1-x^2}}+x\frac{-2x}{2\sqrt{1-x^2}}+\sqrt{1-x^2}\right)=\frac{1}{2\sqrt{1-x^2}}\left(1-x^2+1-x^2\right)=\frac{1-x^2}{\sqrt{1-x^2}}$$

$$=\sqrt{1-x^2}. \checkmark$$