Math 142, Exam 1, Fall 2009 Solutions

Write your answers as legibly as you can.

There are 11 problems on 6 pages. Problem 1 is worth 10 points. Each of the other problems is worth 9 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website a few hours after the exam is finished.

1. **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

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Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a, b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum\limits_{i=1}^{n} f(x_i^*)\Delta_i$.

2. **State both parts of the Fundamental Theorem of Calculus. Be sure to explain all of your notation.**

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Let f be a continuous function defined on the closed interval $[a, b]$.

(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a, b]$, then $A'(x) = f(x)$ for all $x \in [a, b]$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. **Find** $\int 5x\cos(x^2)dx$. **Check your answer.**

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Let $u = x^2$. So, $du = 2xdx$. The original integral is equal to

$$\frac{5}{2}\int \cos u du = \frac{5}{2}\sin u + C = \boxed{\frac{5}{2}\sin(x^2) + C.}$$

Check: The derivative of the proposed answer is

$$\frac{5}{2}2x\cos(x^2) \checkmark.$$

4. **Find** $\displaystyle\int \frac{y^2 dy}{\sqrt{3-4y}}$ **. Check your answer.**

We do a change of variables. Suppose that the expression under the radical was just a variable, then we could do algebraic tricks to finish the problem! We make it so! Let $u = 3-4y$. It follows that $du = -4dy$. The original problem is equal to

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$$\int \frac{\left(\frac{3-u}{4}\right)^2 du}{-4\sqrt{u}} = \frac{1}{-64}\int \frac{(3-u)^2 du}{\sqrt{u}}$$

$$= \frac{1}{-64}\int \frac{(9-6u+u^2)du}{\sqrt{u}} = \frac{1}{-64}\int (9u^{-1/2} - 6u^{1/2} + u^{3/2})du$$

$$= \frac{1}{-64}\left(9u^{1/2}2 - 6u^{3/2}\frac{2}{3} + u^{5/2}\frac{2}{5}\right) + C$$

$$= \boxed{\frac{1}{-64}\left(9(3-4y)^{1/2}2 - 6(3-4y)^{3/2}\frac{2}{3} + (3-4y)^{5/2}\frac{2}{5}\right) + C}$$

Check: The derivative of the proposed answer is

$$\frac{1}{-64}\left(9(3-4y)^{-1/2}(-4) - 6(3-4y)^{1/2}(-4) + (3-4y)^{3/2}(-4)\right)$$

$$= \frac{(3-4y)^{-1/2}(-4)}{-64}\left(9 - 6(3-4y) + (3-4y)^2\right)$$

$$= \frac{1}{16\sqrt{3-4y}}\left(9 - 18 + 24y + 9 - 24y + 16y^2\right) = \frac{1}{16\sqrt{3-4y}}\left(16y^2\right) = \frac{y^2}{\sqrt{3-4y}} \checkmark.$$

5. **Find** $\displaystyle\int \frac{x}{\sqrt{1-4x^4}}dx$ **. Check your answer.**

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We plan to maneuver the given integral into the form

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C.$$

Let $u = 2x^2$. It follows that $du = 4xdx$. The original problem is equal to

$$\frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\arcsin u + C = \boxed{\frac{1}{4}\arcsin(2x^2) + C.}$$

Check: The derivative of the proposed answer is

$$\frac{1}{4}4x\frac{1}{\sqrt{1-(2x^2)^2}} \checkmark.$$

6. **Find the area between $x + y = 0$ and $2 = x + y^2$.**

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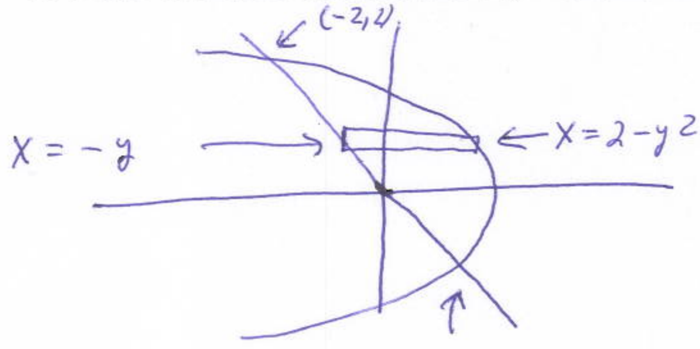

The area is

$$\int_{-1}^{2} 2 - y^2 - (-y)dy = 2y - \frac{y^3}{3} + \frac{y^2}{2}\Bigg|_{-1}^{2} = \boxed{4 - \frac{8}{3} + 2 - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)}.$$

7. **Find the length of of the curve parameterized by $x = \cos 2t$, $y = \sin 2t$, with $0 \le t \le \pi/2$.**

The curve in question is a semi-circle of radius 1; so the length is $\boxed{\pi}$. Also, the length is

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$$\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi/2} \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{4(\sin^2 2t + \cos^2 2t)}\, dt = \int_0^{\pi/2} \sqrt{4}\, dt = 2t\Big|_0^{\pi/2} = \boxed{\pi.}$$

8. **Consider the region in the first quadrant bounded by $y = x^2$ and $x = 1$. Revolve this region about $x = 4$. Find the volume of the resulting solid?**

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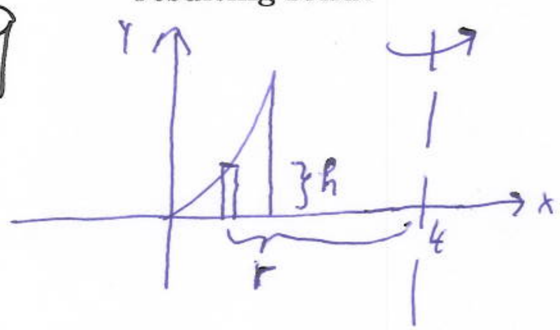

Spin the rectangle. Get a shell of volume $2\pi rht$, where $t = dx$, $r = 4 - x$,

$h = x^2$. The volume of the solid is:

$$2\pi \int_0^1 (4-x)x^2 dx = 2\pi \int_0^1 \left(4x^2 - x^3\right) dx = 2\pi \left(\frac{4x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \boxed{2\pi \left(\frac{4}{3} - \frac{1}{4}\right)}.$$

9. **Consider the region in the first quadrant bounded by $y = e^x$ and $x = \ln 3$. Revolve this region about the x-axis. Find the volume of the resulting solid.**

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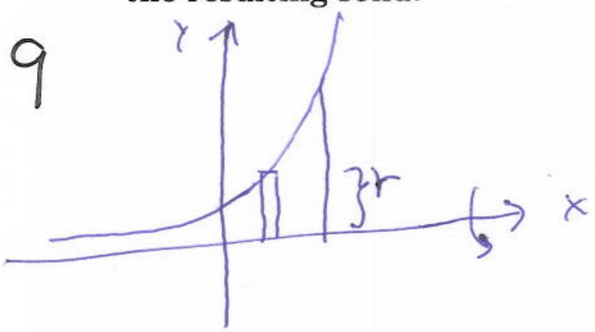

Spin the rectangle. Get a disk of volume $\pi r^2 t$ where $r = e^x$ and $t = dx$. The volume of the solid is

$$\pi \int_0^{\ln 3} e^{2x} dx = \pi \frac{e^{2x}}{2}\Bigg|_0^{\ln 3} = \pi \left(\frac{e^{2\ln 3}}{2} - \frac{1}{2}\right) = \pi \left(\frac{9}{2} - \frac{1}{2}\right) = \boxed{4\pi}.$$

10. **Consider the solid whose base in the first quadrant of the xy plane is bounded by $y = x^2$ and $y = 1$. Each cross section of this solid perpendicular to the y-axis is a square. Find the volume of the resulting solid.**

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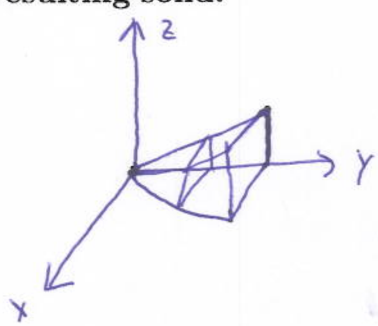

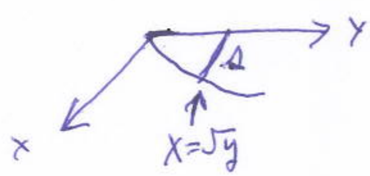

The slice with y-coordinate y has volume side $^2 t$ where $t = dy$ and the side is the x-coordinate of the point on $y = x^2$ where the y-coordinate is y. So the side is $\sqrt{y}$. The volume of our slice is $\sqrt{y}^2 dy = ydy$. The volume of the solid is

$$\int_0^1 ydy = \frac{y^2}{2}\Big|_0^1 = \boxed{1/2}.$$

11. **Suppose that a conical tank is filled with oil which has a density of 50 lb/ft ^{3}. The radius at the top of the tank is 5 ft and the tank is 15 ft high. How much work is done in pumping the oil over the edge of the tank?**

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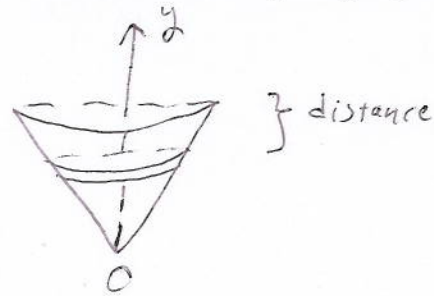
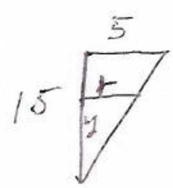


Be sure to draw and label an axis. We have chosen the y-axis to point upward with $y = 0$ being the bottom of the tank. The work to lift the layer of oil with y-coordinate y is

Force $\cdot$ distance $=$ weight $\cdot$ distance $=$ volume $\cdot$ density $\cdot$ distance $= \pi r^2 t \cdot 50 \cdot$ distance,

where $t = dy$, r is computed from similar triangles $\frac{r}{y} = \frac{5}{15}$:



So, $r = \frac{1}{3}y$. The distance to lift the layer of oil whose y-coordinate is y is $15 - y$. The work to lift one layer of oil is:

$$\pi(\tfrac{1}{3}y)^2 50(15 - y)dy.$$

The work to lift all of the oil is

$$\frac{50\pi}{9}\int_0^{15}(15y^2 - y^3)dy = \frac{50\pi}{9}\left(\frac{15y^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^{15} = \frac{50(15)^4\pi}{9}\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \boxed{\frac{50(15)^4\pi}{9\cdot 12}\text{ft-lb.}}$$