Math 142, Exam 1, Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems. Each problem is worth 10 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Find** $\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \sec^2 3\theta d\theta$ **.**

The integral is

$$\frac{\tan 3\theta}{3}\Bigg|_{\frac{\pi}{12}}^{\frac{\pi}{9}} = \frac{\tan\frac{\pi}{3}}{3} - \frac{\tan\frac{\pi}{4}}{3} = \boxed{\tfrac{1}{3}(\sqrt{3}-1)}.$$

2. **Find** $\int_{-1}^{1} \frac{x^2 dx}{\sqrt{x^3+9}}$ **.**

The integral is

$$\frac{2\sqrt{x^3+9}}{3}\Bigg|_{-1}^{1} = \boxed{\frac{2\sqrt{10}}{3} - \frac{2\sqrt{8}}{3}.}$$

3. **Find** $\int_{0}^{\frac{\sqrt{\pi}}{2}} 5x\cos(x^2)dx$ **.**

The integral is

$$\frac{5}{2}\sin(x^2)\Bigg|_{0}^{\frac{\sqrt{\pi}}{2}} = \frac{5}{2}\left(\sin(\frac{\pi}{4}) - \sin 0\right) = \boxed{\frac{5\sqrt{2}}{4}.}$$

4. **State BOTH parts of the Fundamental Theorem of Calculus.**

Let f be a continuous function defined on the closed interval $[a,b]$.

(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a,b]$, then $A'(x) = f(x)$ for all $x \in [a,b]$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

5. **Find** $\lim\limits_{x\to\infty}(1-\frac{1}{2x})^x$ **.**

I know that $\lim\limits_{x\to\infty}(1+\frac{r}{x})^x = e^r$. Apply this fact with $r=-\frac{1}{2}$ to see that the limit is $\boxed{e^{\frac{-1}{2}}}$. (There are other ways to do the problem which require less knowledge and more calculation.)

6. **Find ALL of the area between** $y=x^3$ **and** $y=x$ **.**

I drew a picture. I see that $x^3\ge x$ for $-1\le x\le 0$ and $x\ge x^3$ for $0\le x\le 1$. It follows that the area is

$$\int_{-1}^{0}(x^3-x)dx+\int_0^1(x-x^3)dx = (\tfrac{x^4}{4}-\tfrac{x^2}{2})\Big|_{-1}^{0}+(\tfrac{x^2}{2}-\tfrac{x^4}{4})\Big|_{0}^{1}$$

$$= -(\tfrac{1}{4}-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{4}) = \boxed{\tfrac{1}{2}}.$$

7. **Find the length of** $x=\frac{1}{8y^2}+\frac{y^4}{4}$ **from** $y=1$ **to** $y=2$ **.**

The length is

$$\int_1^2\sqrt{1+(\frac{dx}{dy})^2}\,dy = \int_1^2\sqrt{1+(\frac{-1}{4y^3}+y^3)^2}\,dy = \int_1^2\sqrt{1+\frac{1}{16y^6}-\frac{1}{2}+y^6}\,dy$$

$$= \int_1^2\sqrt{\frac{1}{16y^6}+\frac{1}{2}+y^6}\,dy = \int_1^2\sqrt{(\frac{1}{4y^3}+y^3)^2}\,dy = \int_1^2\frac{1}{4y^3}+y^3\,dy = \frac{-1}{8y^2}+\frac{y^4}{4}\Big|_1^2$$

$$= \boxed{\tfrac{-1}{32}+4-(-\tfrac{1}{8}+\tfrac{1}{4}).}$$

8. **Consider the region bounded by** $y=2x^2$ **and** $y=2\sqrt{x}$**. Revolve this region about** $x=5$**. Set up an integral that gives the volume of the resulting solid? You do not have to evaluate the integral.**

I drew a picture. I use shells. Chop the x-axis into subintervals. Draw a rectangle over each subinterval. Spin the rectangle. Get a shell of volume $2\pi rht$, where $t=dx$, $r=5-x$, $h=2\sqrt{x}-2x^2$. Add up the volume inside these shells. Take the limit. In other words, the volume is

$$\boxed{2\pi\int_0^1(5-x)(2\sqrt{x}-2x^2)dx.}$$

9. **Consider the region bounded by $y = 2x^2$ and $y = 2\sqrt{x}$. Revolve this region about $y = -5$. Set up an integral that gives the volume of the resulting solid? You do not have to evaluate the integral.**

I drew a picture. I use shells. Chop the y-axis into subintervals. Draw a rectangle over each subinterval. Spin the rectangle. Get a shell of volume $2\pi rht$, where $t = dy$, $r = y + 5$, $h = \sqrt{\frac{y}{2}} - \frac{y^2}{4}$. Add up the volume inside these shells. Take the limit. In other words, the volume is

$$\boxed{2\pi \int_0^2 (y+5)\left(\sqrt{\frac{y}{2}} - \frac{y^2}{4}\right)dy.}$$

10. **Consider the solid whose base in the xy plane is bounded by $y = x^2$, $y = 0$, and $x = 1$. Each cross section of this solid perpendicular to the x-axis is an equilateral triangle. Set up an integral that gives the volume of the resulting solid? You do not have to evaluate the integral.**

Slice the solid. Each slice has thickness dx. The base of the slice with x-coordinate x goes from the x-axis to $y = x^2$. This base is x^2 units long. The slice with base x^2 and thickness dx is an equilateral triangle so the height of the slice is $\frac{\sqrt{3}}{2}x^2$. The volume of the slice is one half the base times the height times the thickness or $\frac{1}{2}x^2\frac{\sqrt{3}}{2}x^2dx$. Add up the volume of all of these slices. Take the limit. In other words, integrate. The volume of the solid is

$$\boxed{\frac{\sqrt{3}}{4}\int_0^1 x^4 dx.}$$